AlterMundus

Alain Matthes

Collection AlterMundus

Sangaku Gallery

tkz-2d.sty v3.1 28 01 2008

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tkz-2d.sty v3.1

*This document is a litte gallery of some simple sangaku. I made all these pictures with my package **tkz-2d.sty**.*

[sangaku v1.0 28 01 2008]

- Firstly, I would like to thank **Till Tantau** for the beautiful LATEX package, namely TikZ.
- I am grateful to **Michel Bovani** for providing the `fourier` font.
- I received much valuable advice from **Jean-Côme Charpentier** and **Josselin Noirel**.

Contents

Gallery : Some Sangaku Problems

Some references

- a Scientific American article by Tony Rothman written in co-operation with Hidetoshi Fukagawa ;
- the book by H. Fukagawa and D. Pedoe.
- http://www.sangaku.info/
- http://mathworld.wolfram.com
- http://www.wasan.jp/english/
- http://www.cut-the-knot.org/pythagoras/Sangaku.shtml

What are Sangaku or San Gaku ?

Sangaku are colorful wooden tablets which were hung often in shinto shrines and sometimes in buddhist temples in Japan and posing typical and elegant mathematical problems. The problems featured on the sangaku are problems of japanese mathematics (wasan). The earliest sangaku found date back to the beginning of the 17th century.

Sangaku Two Unrelated Circles

Chord [ST] *is perpendicular to diameter* [CP] *of a circle with center* O *at point* R. Q *is point of* [CP] *between* P *and* R . [SQ] *intersects the circle in* V.
Let r *be the radius of the circle inscribed into the curvilinear triangle TQV. Prove that*

$$\frac{1}{r} = \frac{1}{\mathrm{PQ}} + \frac{1}{\mathrm{QR}}$$

Example n° 1 Two Unrelated Circles

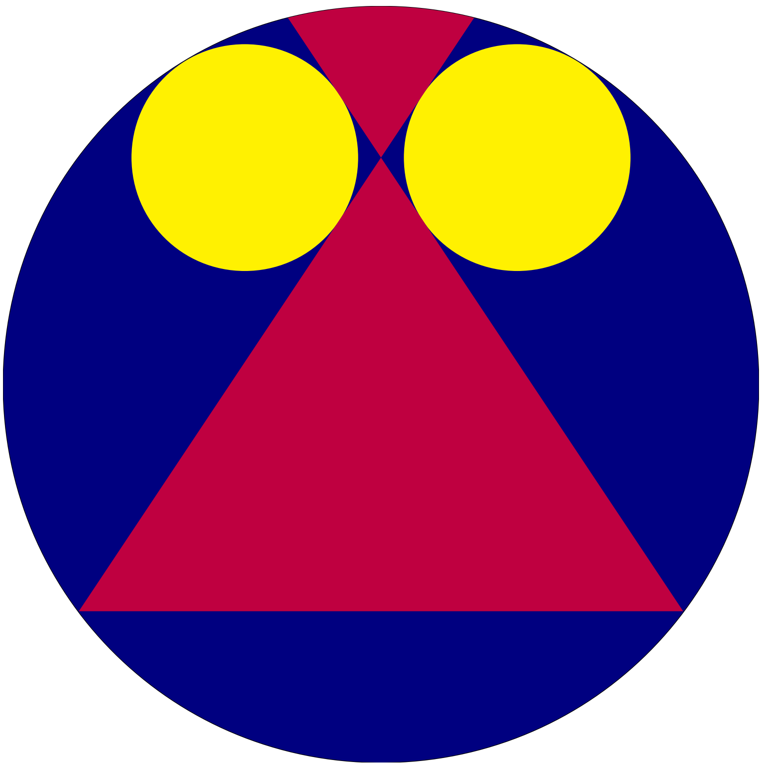
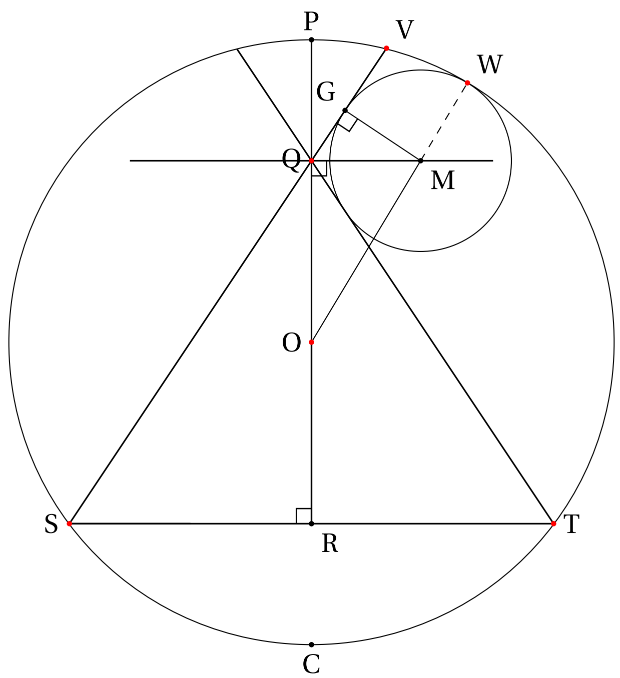


FIG. 1: Sangaku Two Unrelated Circles

Now you can read the code to get the last picture.

```
\begin{tikzpicture}
    \tkzInit[xmin = -5,ymin = -5,xmax = 5,ymax = 5]
    \tkzPoint*(0,0){O}\tkzPoint*(-2,-3){A}
    \tkzPoint*(2,-3){B}\tkzPoint*(0,3){Q}
    \tkzCircleR(O,5 cm)
    \tkzInterLCR(A,B)(O,5 cm){S}{T}
    \tkzInterLCR(S,Q)(O,5 cm){V}{X}
    \tkzClipCircle(O,5 cm)
    \tkzLine[prefix = dm](S/Q)\tkzLine[prefix = dn](T/Q)
    \tkzMidPoint*(S,T){R}
    \tkzLineOrth(O,Q)(Q)
    \tkzMathLength(R,Q)\let\dRQ\tkzmathLen%
    \tkzMathLength(S,Q)\let\dSQ\tkzmathLen%
    \pgfmathparse{\dSQ/\dRQ*1.5}
    \tkzPoint*(\pgfmathresult,3){M}\tkzPoint*(-\pgfmathresult,3){N}
    \tkzProjection*[pos = above left](V,Q)(M/G)
    \tkzInterLCR(O,M)(O,5 cm){W}{Y}
    \tkzLine(A/B,O/Q,S/Q,T/Q)
    \tkzCircleR(M,1.5 cm)
    \tkzFillCircle[color = blue!50!Black](O,5cm)
    \tkzFillCircle[color = yellow](M,1.5cm)
    \tkzFillCircle[color = yellow](N,1.5cm)
    \tkzFillPolygon[color = purple](S,T,Q)
    \tkzFillPolygon[color = purple](dmr,dnr,Q)
\end{tikzpicture}
```

Explanation :



Sangaku - Solution of Nathan Bowler

Take coordinates such that it is the unit circle ($r = 1$), with Q on the x axis.
Let G = $(a\,;\,b)$, W = $(u\,;\,v)$.
Then :

- The line (QV) has equation $ax + by = 1$,

- Q is at $\left(\frac{1}{a};0\right)$,
- O is at $\left(\frac{1}{a};\frac{v}{au}\right)$.
- V satisfies

$$ax+by=1 \text{ and } \left(x-\frac{1}{a}\right)^2+\left(y-\frac{v}{au}\right)^2=\left(1-\frac{1}{au}\right)^2.$$

Further

$$y^2-\frac{av}{u}2y+2\frac{a}{u}-a^2-1=0$$

so that

$$\begin{aligned}\mathrm{QR}&=-y\\&=-\frac{av}{u}-\sqrt{\left(\frac{a^2}{u^2}-2\frac{a}{u}+1\right)}\\&=-\frac{av}{u}+1-\frac{a}{u}\\&=1-\frac{a(1+v)}{u}.\end{aligned}$$

and

$$\mathrm{PQ}=1-\frac{1}{au}+\frac{v}{au}=1-\frac{1-v}{au}.$$

So

$$(\mathrm{PQ}-1)(\mathrm{QR}-1)=\frac{(1-v)(1+v)}{u^2}=1.$$

Equivalently,

$$\frac{1}{\mathrm{PQ}}+\frac{1}{\mathrm{QR}}=1=\frac{1}{r}.$$

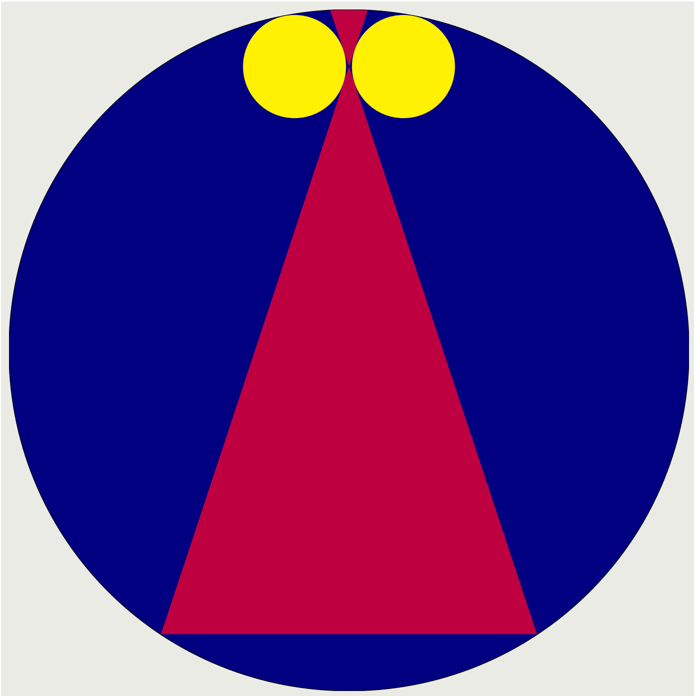
FIG. 2: Sangaku-Two Unrelated Circles : R = 6 cm, OR = −2 cm et RQ = 4 cm

```
\begin{tikzpicture}[scale = .7]
   \TwoUnrelatedCircles{6}{-2}{4}
\end{tikzpicture}
```

In the general case, some information will be needed. For instance, it is necessary to give the radius of the big circle, the position of points R and Q. I have decided to give R$(0, r)$ relatively to O and Q relatively to P with the value of PQ.
The macro, below, is used to obtain some examples.

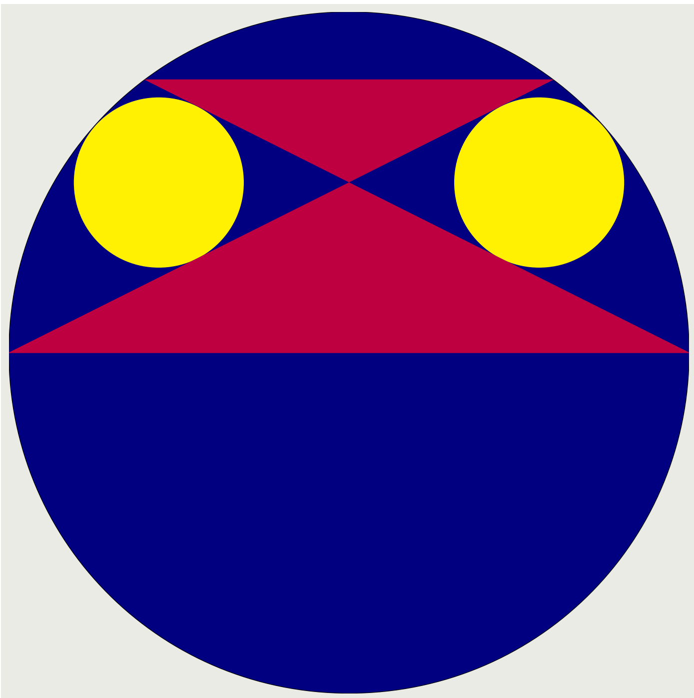
```
\newcommand*{\TwoUnrelatedCircles}[3]{
 % #1 --> R ; #2 --> Y_P <0 or >0 ; #3 --> PQ > =  0
\xdef\ORadius{#1}
\xdef\tucR{#2}
\xdef\tucQR{#3}
\pgfmathparse{\tucQR+\tucR}
\xdef\tucQ{\pgfmathresult}
\pgfmathparse{\ORadius-(\tucQR+\tucR)}
\xdef\tucPQ{\pgfmathresult}
\pgfmathparse{(\tucPQ*\tucQR)/(\tucPQ+\tucQR)}
\let\tucr\pgfmathresult%
\pgfmathparse{\ORadius-\tucr}
\let\OORadius\pgfmathresult%
\tkzInit[xmin = -\ORadius,ymin = -\ORadius,%
         xmax = \ORadius, ymax = \ORadius]\tkzClip
\tkzPoint(0,0){O}
\tkzPoint*(-\ORadius,\tucR){A}
\tkzPoint*(\ORadius,\tucR){B}
\tkzPoint(0,\tucQ){Q}
\tkzCircleR(O,\ORadius cm)
\tkzInterLCR(A,B)(O,\ORadius cm){S}{T}
\tkzInterLCR(S,Q)(O,\ORadius cm){S}{V}
\tkzInterLCR(Q,T)(O,\ORadius cm){Y}{T}
\tkzClipCircle(O,\ORadius cm)
\tkzLine[prefix = dm,kl = 3,kr = 5](Q/V)
\tkzLine[prefix = dn,kl = 3,kr = 5](Q/Y)
\tkzPoint(0,\tucR){R}
\tkzPoint(-\ORadius,\tucQ){U}
\tkzInterLCR(U,Q)(O,\OORadius cm){M}{N}
\tkzLine(S/Q,T/Q)
\tkzCircleR(M,\tucr cm)
\tkzFillCircle[color = blue!50!Black](O,\ORadius cm)
\tkzFillCircle[color = purple](M,\tucr cm)
\tkzFillCircle[color = purple](N,\tucr cm)
\tkzFillPolygon[color = orange](S,T,Q)
\tkzFillPolygon[color = orange](V,Y,Q)
}%
```

New examples



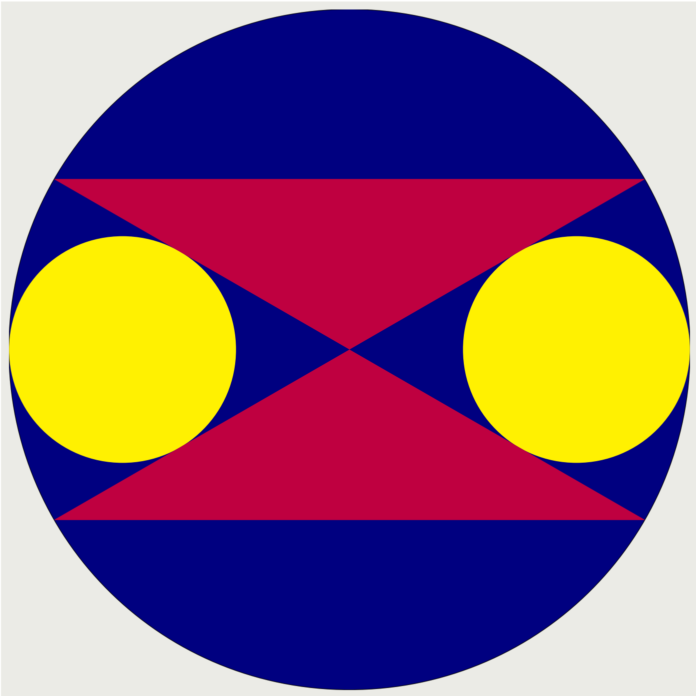
```
\begin{tikzpicture}[scale = .75]
   \TwoUnrelatedCircles{6}{-5}{10}
 \end{tikzpicture}
```

FIG. 3: Sangaku-Two Unrelated Circles : R = 6 cm, OR = −5 cm et RQ = 10 cm



```
\begin{tikzpicture}[scale = .75]
   \TwoUnrelatedCircles{6}{0}{3}
 \end{tikzpicture}
```

FIG. 4: Sangaku-Two Unrelated Circles : R = 6 cm, OR = 0 cm et RQ = 3 cm



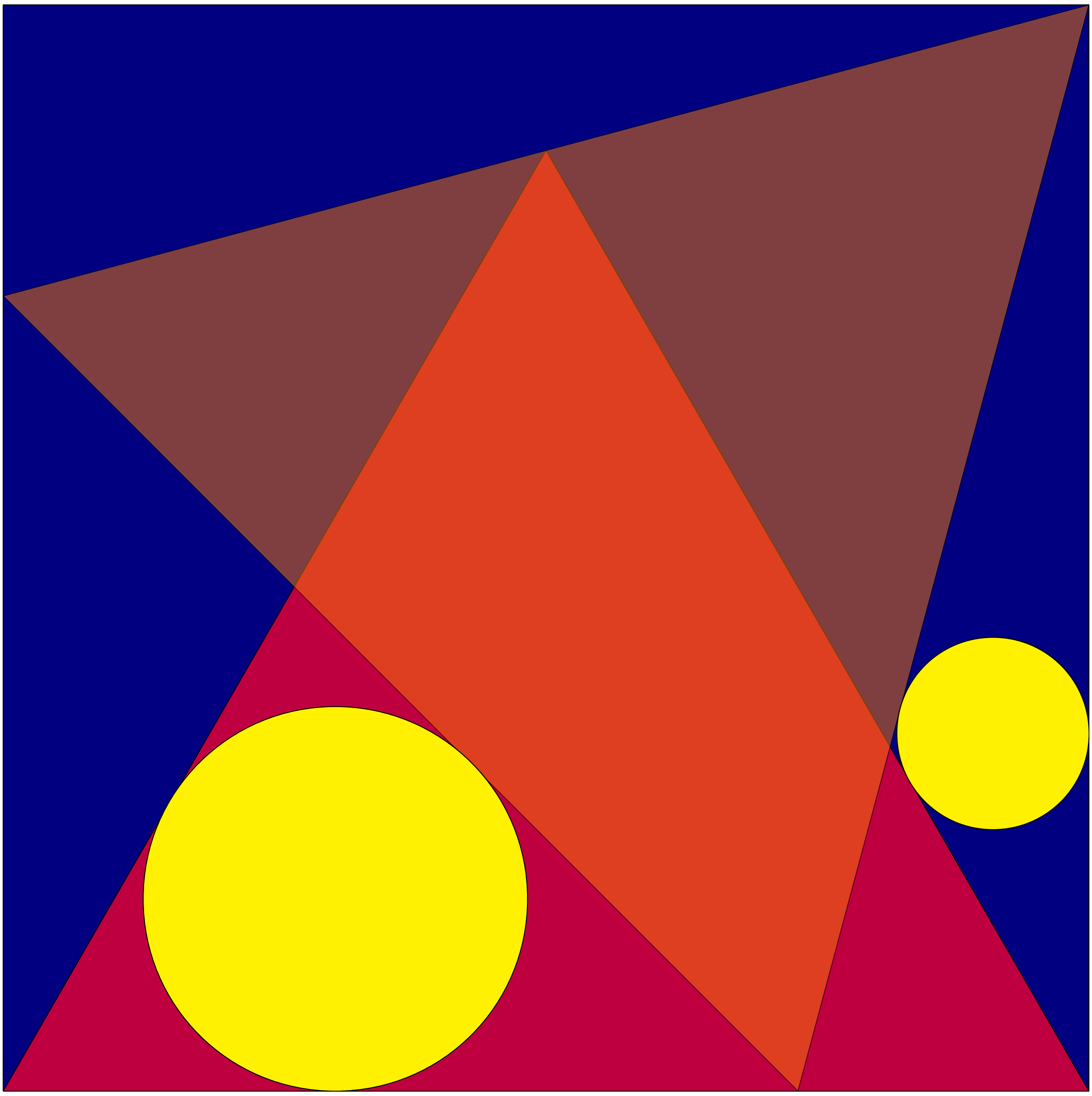
```
\begin{tikzpicture}[scale = .75]
   \TwoUnrelatedCircles{6}{-3}{3}
 \end{tikzpicture}
```

FIG. 5: Sangaku-Two Unrelated Circles : R = 6 cm, OR = −3 cm et RQ = 3 cm

Sangaku in a square - I

Here is an elegant sangaku that requires both geometric and algebraic skills and some perseverance : Two equilateral triangles are inscribed into a square as shown in the diagram. Their side lines cut the square into a quadrilateral and a few triangles. Find a relationship between the radii of the two incircles shown in the diagram.

Example n° 2 Sangaku in a square - I



The code of the last figure is

```
\begin{tikzpicture}[scale = 1.75]
    \tkzInit
    \tkzPoint*(0,0){B} \tkzPoint*(8,0){C}%
    \tkzPoint*(0,8){A} \tkzPoint*(8,8){D}
    \tkzSquare*(B,C){D}{A}
    \tkzPolygon(B,C,D,A)\path[clip] (B)--(C)--(D)--(A)--cycle;
    \draw[fill = blue!50!black] (B)--(C)--(D)--(A)--cycle;
    \tkzTrEqui*(B,C){M}
    \draw[fill = red] (B)--(C)--(M)--cycle;
    \tkzInterLL*(D,M)(A,B){N}
    \tkzRotate*(N,-60)(D/L)
    \tkzBisector*(M,B,C){x}
    \tkzBisector*(N,L,B){y}
    \tkzInterLL*(L,y)(B,x){H}
    \tkzBisector*(M,C,D){u}
    \tkzBisector*(L,D,C){v}
    \tkzProjection*(C,B)(H/I)
    \draw[fill = orange,opacity = .5] (D)--(N)--(L)--cycle;
    \tkzCircle[style = {fill = yellow}](H,I)
    \tkzInterLL*(C,u)(D,v){K}
    \tkzProjection*(C,D)(K/J)
    \tkzProjection*(B,C)(M/E)
    \tkzCircle[style = {fill = yellow}](K,J)
\end{tikzpicture}
```

Firstly, we need to prove that it is possible that two equilateral triangles are inscribed into a square as shown in the diagram. A theorem exists but it is nice to find a solution in this particuler case. Let ABCD a square, BCM an equilateral triangle. The line (DM) intersects [AB] at point N. Then we construct a point L on the side [BC] and the angle $\widehat{\text{NDL}} = 60°$

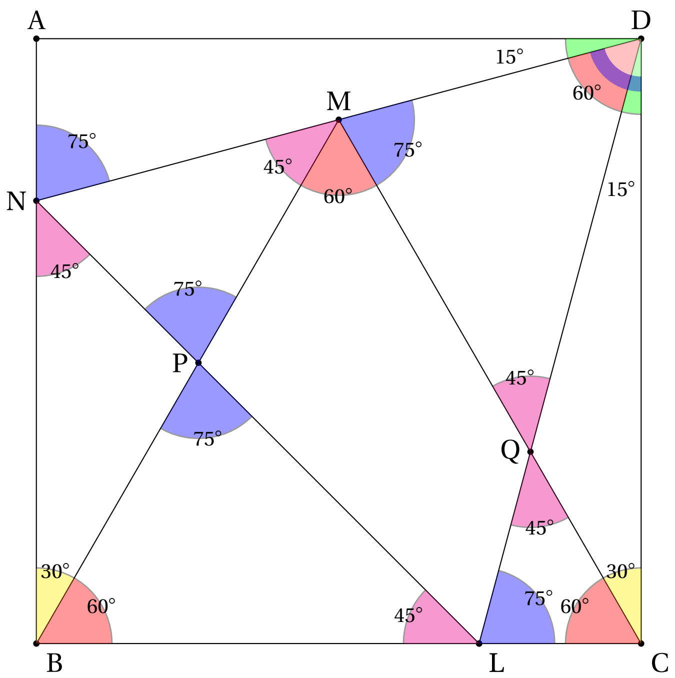
The triangle MCD is an isosceles triangle with two sides MC and CD of the same length a. It follows that

$\widehat{\text{MDC}} = \widehat{\text{DMC}} = 75°$ because $\widehat{\text{MCD}} = 30°$

Now we we can determine the angular size of all the angles

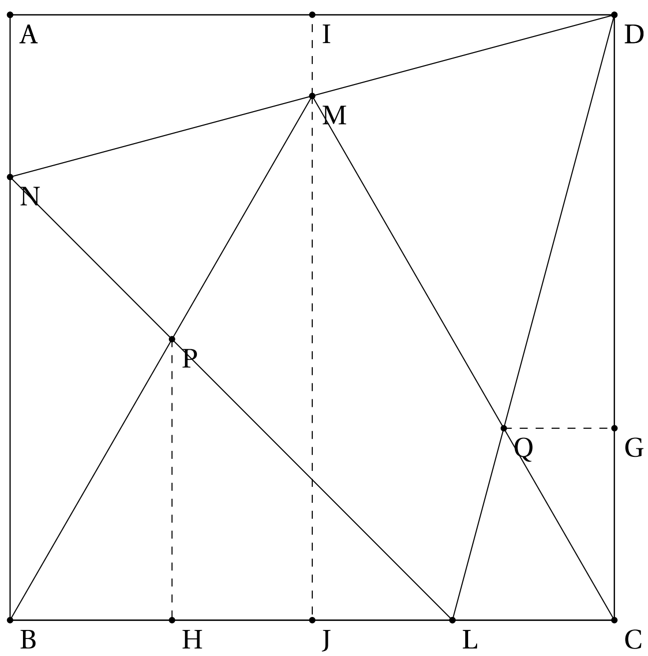
$\widehat{\text{LDC}} = 15°$ so $\widehat{\text{ADN}} = 15°$ and $\widehat{\text{AND}} = 75°$

AN = LC then BL = BN and $\widehat{\text{BLN}} = \widehat{\text{LNB}} = \widehat{\text{NMB}} = \widehat{\text{MQD}} = \widehat{\text{LQC}} = 45°$



```
\begin{tikzpicture}
     \tkzInit
     \tkzPoint(0,0){B}
     \tkzPoint(8,0){C}
     \tkzPoint*(0,8){A}
     \tkzPoint*(8,8){D}
     \tkzPolygon(B,C,D,A)
     \tkzTrEqui*(B,C){M}
     \tkzDrawPoint[pos = above](M,A,D)
     \tkzInterLL*(D,M)(A,B){N}
     \tkzRotate(N,-60)(D/L)
     \tkzDrawPoint(L)
     \tkzSegment(D/N,N/L,L/D,B/M,M/C)
     \tkzInterLL*(N,L)(M,B){P}
     \tkzInterLL*(M,C)(D,L){Q}
     \tkzDrawPoint[pos = left](N,P,Q)
     \tkzMarkAngle[fillcolor = green,dist = .40,label = 15^{\circ}](L/D/C)
     \tkzMarkAngle[fillcolor = green,dist = .35,label = 15^{\circ}](A/D/N)
     \tkzMarkAngle[fillcolor = yellow,dist = .20,label = 30^{\circ}](M/C/D,A/B/M)
     \tkzMarkAngle[fillcolor = magenta,dist = .20,label = 45^{\circ}]%
                  (B/L/N,L/N/B,N/M/B,M/Q/D,L/Q/C)
     \tkzMarkAngle[fillcolor = blue,dist = .20,label = 75^{\circ}]%
                  (A/N/D,N/P/M,B/P/L,C/L/D,C/M/D)
     \tkzMarkAngle[fillcolor = red,dist = .20,label = 60^{\circ}]%
                  (M/B/C,C/M/B,M/C/B,N/D/L)
     \tikzstyle{ai} = [draw,line width = .2cm]
     \tkzMarkAngle[size = .6,color = blue](C/D/M)
\end{tikzpicture}
```

We can see that the angles $\widehat{\mathrm{DNL}}$ and $\widehat{\mathrm{NLD}}$ have the same degree measurements 60°. DNL is an equilateral triangle and it is the largest equilateral triangle which can be inscribed in the square (Madachy 1979). We prove lately that the side is $s = (\sqrt{6} - \sqrt{2})a$.



```
\begin{tikzpicture}
    \tkzInit[xmin = -1,ymin = -1]\tkzClip
    \tkzPoint(0,0){B}
    \tkzPoint(8,0){C}
    \tkzPoint(0,8){A}
    \tkzPoint(8,8){D}
    \tkzSquare(B,C){D}{A}
    \tkzPolygon(B,C,D,A)
    \tkzTrEqui(B,C){M}
    \tkzInterLL(D,M)(A,B){N}
    \tkzRotate(N,-60)(D/L)
    \tkzDrawPoint(L)
    \tkzSegment(D/N,N/L,L/D)
    \tkzInterLL(N,L)(M,B){P}
    \tkzInterLL(M,C)(D,L){Q}
    \tkzProjection(B,C)(P/H)
    \tkzProjection(C,D)(Q/G)
    \tkzProjection(B,C)(M/J)
    \tkzProjection(D,A)(M/I)
    \tkzSegment[style = dashed](M/I,M/J,P/H,Q/G)
\end{tikzpicture}
```

We need some preliminaries to find the ratio between the radii of the two incircles shown in the first diagram.
Assume the side of the square equals a
First, we determine MI

$$\mathrm{MJ} = \frac{\sqrt{3}}{2}a \text{ and } \mathrm{MI} = a - \frac{\sqrt{3}}{2}a = \frac{(2-\sqrt{3})}{2}a$$

Thus we can find AN and NB

$$\mathrm{AN} = 2\mathrm{MI} = (2-\sqrt{3})a$$

and

$$\mathrm{BN} = \mathrm{AB} - \mathrm{AN} = a - (2-\sqrt{3})a = (\sqrt{3}-1)a$$

ADN is a right triangle with hypotenuse ND. We have, $\mathrm{AD}^2 + \mathrm{AN}^2 = \mathrm{ND}^2$ by the Pythagorean theorem. Using this, we continue :

$$\mathrm{ND}^2 = a^2 + (2-\sqrt{3})^2 a^2 = a^2(8-4\sqrt{3})$$

$$\mathrm{ND} = \mathrm{NL} = \mathrm{LD} = (\sqrt{6}-\sqrt{2})a$$

The value of $\tan(15^\circ)$ which will be useful later on.

$$\tan 15^\circ = \frac{\mathrm{AN}}{\mathrm{AD}} = \frac{(2-\sqrt{3})a}{a} = 2-\sqrt{3}$$

Now we can apply the standard formula in a triangle to determine the inradius

$$r = \frac{sh}{p}$$

where r, p, s, h are respectively the inradius, perimeter, a side and the altitude to the side in a triangle.
A good idea is to find a relationship between p and h
Let BPL the first triangle, here $h = \mathrm{PH}$, $l = \mathrm{BL}$ and $p = \mathrm{BP} + \mathrm{PL} + \mathrm{LB}$.

$$\sin(60^\circ) = \frac{\sqrt{3}}{2} = \frac{\mathrm{PH}}{\mathrm{BP}} = \frac{h}{\mathrm{BP}}$$

thus

$$\mathrm{BP} = \frac{2h}{\sqrt{3}}$$

HPL is an isosceles right triangle. The hypotenuse PL has length $\sqrt{2}h$.
And finally the relation between BL and h can be obtain like this

$$\mathrm{BL} = \mathrm{BH} + \mathrm{HL} = \frac{h}{\sqrt{3}} + h$$

In an other way

$$\mathrm{BL} = \mathrm{BN} = (\sqrt{3} - 1)a$$

The inradius r_1 of BLP is

$$r_1 = \frac{sh}{p} = \frac{(\sqrt{3}-1)ah}{\frac{2h}{\sqrt{3}} + \sqrt{2}h + \frac{h}{\sqrt{3}} + h} = \frac{(\sqrt{3}-1)a}{1+\sqrt{2}+\sqrt{3}}$$

For CQD, similarly the inradius r_2 can be found with

$$r_2 = \frac{ah}{p}$$

with $p = \mathrm{CQ} + \mathrm{QD} + \mathrm{DC}$ and $h = \mathrm{QG}$
or

$$\frac{\mathrm{QD}}{\mathrm{DL}} = \frac{\mathrm{QG}}{\mathrm{LC}} = \frac{h}{(2-\sqrt{3})a}$$

from which

$$\mathrm{QD} = \frac{(\sqrt{6}-\sqrt{2})ah}{(2-\sqrt{3})a} = (\sqrt{6}+\sqrt{2})h$$

We continue

$$\frac{\mathrm{QG}}{\mathrm{QC}} = \frac{1}{2} = \frac{h}{\mathrm{GC}}$$

from which

$$\mathrm{QC} = 2h$$

and finally $\mathrm{DC} = \mathrm{DG} + \mathrm{GC} =$

$$\frac{h}{\mathrm{DG}} = \tan(15^\circ) = 2 - \sqrt{3}$$

from which

$$\mathrm{DG} = \frac{h}{2-\sqrt{3}} = (2+\sqrt{3})h$$

and

$$\frac{\mathrm{GC}}{\mathrm{QC}} = \frac{\mathrm{GC}}{2h} = \frac{\sqrt{3}}{2}$$

Thus we can find

$$\mathrm{GC} = \sqrt{3}h$$

Finally

$$p = (\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h$$

The second radius is

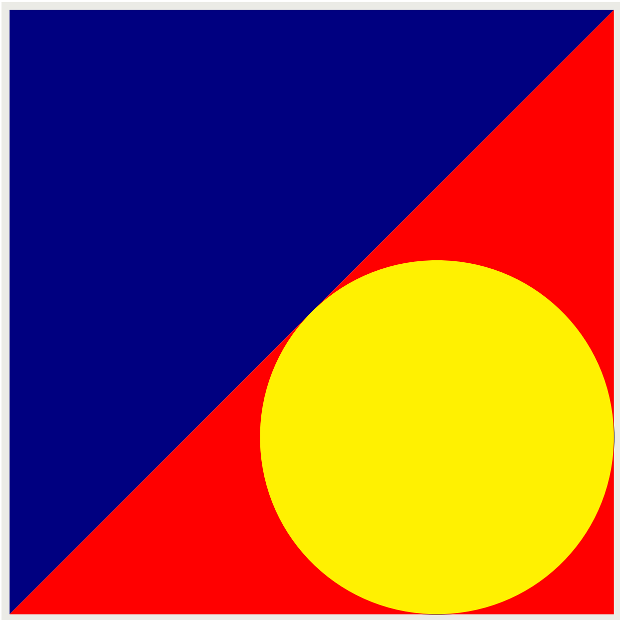
$$r_2 = \frac{ah}{p} = \frac{ah}{(\sqrt{6}+\sqrt{2})h + 2h + (2+\sqrt{3})h + \sqrt{3}h} = \frac{a}{4+\sqrt{2}+2\sqrt{3}+\sqrt{6}}$$

Without a special effort, we conclude that $r_1 = 2r_2$

Sangaku in a square - II

A very simple sangaku.
Find a relationship between the radius of the yellow circle and the side of the square.

Example n° 3 Sangaku in a square - II



```
\begin{tikzpicture}
  \tkzInit
  \tkzPoint*(0,0){A} \tkzPoint*(8,0){B}
  \tkzSquare*(A,B){C}{D}
  \tkzProjection*(A,C)(B/F)
  \tkzBisector*(A,C,B){K}
  \tkzInterLL*(C,K)(B,F){I}
  \tkzCircle(I,F)
  \tkzMathLength(I,F)
  \tkzFillPolygon[color = blue!50!black](A,C,D)%
  \tkzFillPolygon[color = red](A,B,C)%
  \tkzFillCircle[color = yellow](I,\tkzmathLen pt)%
\end{tikzpicture}
```

Firstly, we can found the relationship between the inradius and the sides of a right triangle.
If r is the inradius of a circle inscribed in a right triangle with sides a and b and hypotenuse c, then

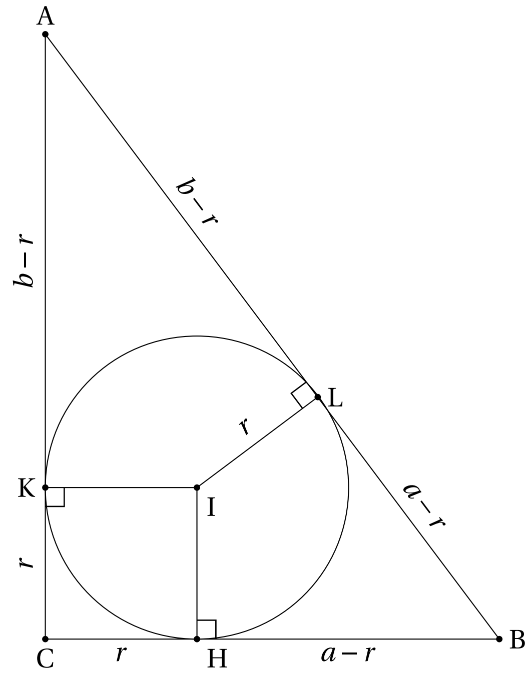
$$r = \frac{1}{2}(a+b-c).$$

Let ABC represents a right triangle, with the right angle located at C, as shown on the figure. Let a, b and c the lengths of the three sides ; c is the length of the hypothenuse.
Let r and p be the radius of the incercle and the semiperimeter of the triangle.
a, b and c can be regarded in relation to r and they may be expressed with r : $a = r+(a-r)$, $b = r+(b-r)$ and $c = (a-r)+(b-r)$.
In a right triangle, we have the relation $r = p/2 - c$. From the diagram, the hypotenuse AB is split in two pieces : $(a-r)$ and $(b-r)$, the length of the hypothenuse is $c = (a-r)+(b-r)$.
The perimeter is a function of r

$$p = a+b+c = r+(a-r)+r+(b-r)+(a-r)+(b-r) = 2a+2b-2r$$

so we can expressed r with s and c

$$2r = a + b - c = p - 2c \text{ and } r = \frac{p}{2} - c = \frac{a+b-c}{2}$$

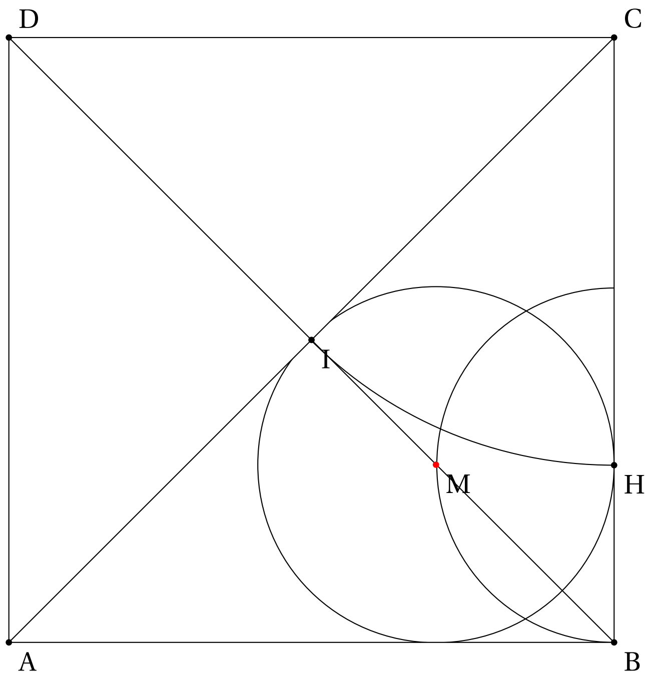



```
\begin{tikzpicture}
     \tkzInit\tkzClip[space = 0.5]
     \tkzPoint[pos = above](0,8){A}
     \tkzPoint[pos = right](6,0){B}
     \tkzPoint[pos = below](0,0){C}
     \tkzPolygon(A,B,C)%%
     \tkzInCenter(A,B,C){I}
     \tkzProjection(B,C)(I/H)
     \tkzProjection[pos = left](A,C)(I/K)
     \tkzProjection[pos = right](A,B)(I/L)
     \tkzSegment(I/L,I/H,I/K)
     \tkzCircle(I,H)
     \tkzRightAngle(A/L/I)
     \tkzRightAngle(B/H/I)
     \tkzRightAngle(C/K/I)
     \tkzSegmentMark[label = $r$,poslabel = -12pt](C/H)
     \tkzSegmentMark[label = $a-r$,poslabel = -12pt](H/B)
     \tkzSegmentMark[label = $r$](C/K)
     \tkzSegmentMark[label = $b-r$](K/A)
     \tkzSegmentMark[label = $a-r$](L/B)
     \tkzSegmentMark[label = $b-r$](A/L)
     \tkzSegmentMark[label = $r$](I/L)
\end{tikzpicture}
```

Now, let ABC represents a isosceles right triangle with AB = AC = a, then AC = $\sqrt{2a}$ and $a+b-c = 2a-\sqrt{2}a$
So the inradius in this case is

$$r = \frac{2-\sqrt{2}}{2}a$$

Now we can obtain the incenter without the bisectors

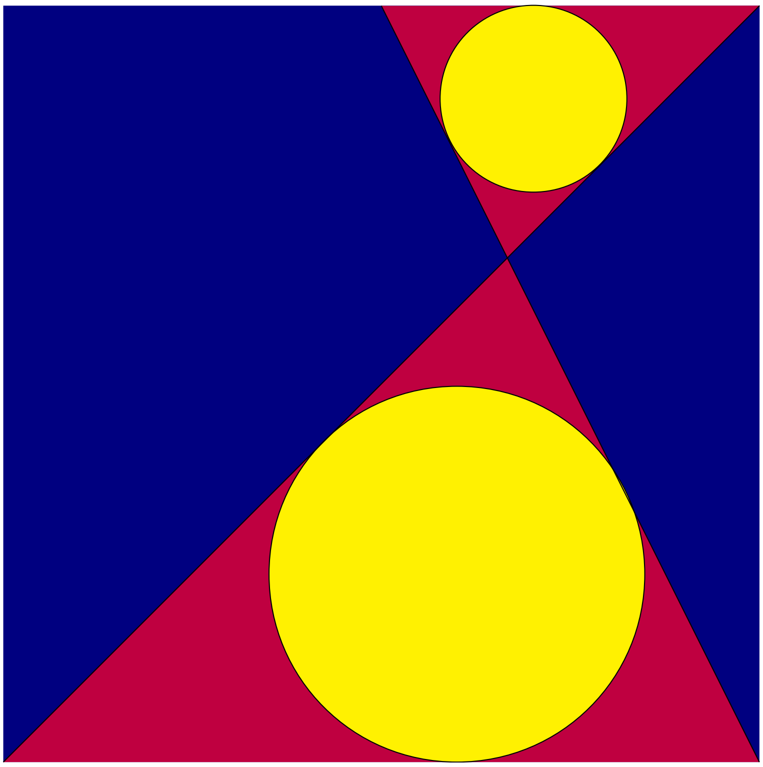



```
\begin{tikzpicture}[scale = 1.25]
  \tkzInit[xmin = -1,ymin = -1,xmax = 9,ymax = 9]
  \tkzClip
  \tkzPoint(0,0){A}
  \tkzPoint(8,0){B}
  \tkzPoint[pos = above right](8,8){C}
  \tkzPoint[pos = above right](0,8){D}
  \tkzSegment(A/B,B/C,C/D,D/A,A/C,B/D)
  \tkzInterLL(A,C)(B,D){I}
  \tkzDuplicateLength(C,I)(C,B){H}
  \tkzClipPolygon(A,B,C)
  \tkzCircle(C,I)
  \tkzCircle(H,B)
  \tkzMathLength(H,B)
  \tkzInterLCR(B,D)%
   (H,\tkzmathLen pt){M}{N}
  \tkzDrawPoint[color = red](M)
  \tkzCircle(M,H)
\end{tikzpicture}
```

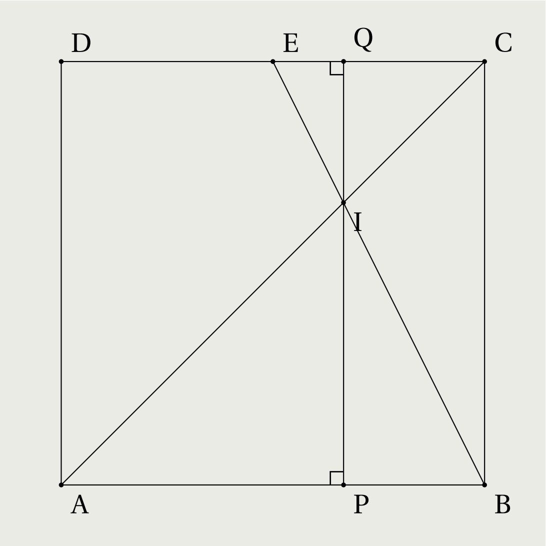
Sangaku in a square - III

In the following diagram, a triangle is formed by a line that joins the base of a square with the midpoint of the opposite side and a diagonal. Find the radius of the two inscribed circles.

Example n° 4 Sangaku in a square - III



```
\begin{tikzpicture}[scale = 1.25]
   \tkzInit\tkzClip
   \tkzPoint*(0,0){A}\tkzPoint*(8,0){B}
   \tkzPoint*(4,8){E}\tkzSquare*(A,B){C}{D}
   \tkzBisector*(C,A,B){I}\tkzBisector*(A,B,E){J}
   \tkzInterLL(A,I)(B,J){K}\tkzProjection*(A,B)(K/H)
   \tkzBisector*(D,C,A){I1}\tkzBisector*(C,E,B){J1}
   \tkzInterLL(C,I1)(E,J1){K1}\tkzProjection*(C,D)(K1/H1)
   \tkzFillPolygon[color = blue!50!black](A,B,C,D)
   \tkzFillPolygon[color = purple](A,C,E,B)
   \tkzCircle[style = {fill = yellow}](K,H)
   \tkzCircle[style = {fill = yellow}](K1,H1)
   \tkzSegment(E/B,A/C)
\end{tikzpicture}
```




```
\begin{tikzpicture}[scale = .7]
  \tkzInit[xmin = -1,ymin = -1,%
           xmax =  9,ymax =  9]
  \tkzClip
  \tkzPoint(0,0){A}
  \tkzPoint(8,0){B}
  \tkzPoint[pos = above right](4,8){E}
  \tkzPoint[pos = above right](8,8){C}
  \tkzPoint[pos = above right](0,8){D}
  \tkzPolygon(A,B,C,D)
  \tkzInterLL(A,C)(B,E){I}
  \tkzProjection(A,B)(I/P)
  \tkzProjection[pos = above right](C,D)(I/Q)
  \tkzRightAngle(A/P/I,D/Q/I)
  \tkzSegment(E/B,A/C,Q/P)
\end{tikzpicture}
```

QC is parallel to the base AB and is half as long which implies that the two triangles QIC and QAB are similar. I divides the segments QP and AD in ratio 2 :1 so that

$$\mathrm{IP} = \frac{2}{3}\mathrm{QP} = \frac{2a}{3}$$

$$\mathrm{AI} = \frac{2}{3}\mathrm{AC}$$

Thus assuming AB = a, we have AC = $\sqrt{2}a$

$$\mathrm{AI} = \frac{2\sqrt{2}}{3}a$$

and

$$\mathrm{BI} = \frac{2}{3}\mathrm{BE}$$

we can apply the Pythagorean theorem to find BE

$$\mathrm{BE} = \frac{\sqrt{5}}{2}a$$

This means that

$$\mathrm{BI} = \frac{\sqrt{5}}{3}a$$

In any triangle,

$$r \times p = s \times h$$

where r is the inradius, p the perimeter, s the side and h the altitude of the triangle.
In other words

$$r\left(1 + \frac{\sqrt{5}}{3} + \frac{2\sqrt{2}}{3}\right)a = a \times \frac{2a}{3}$$

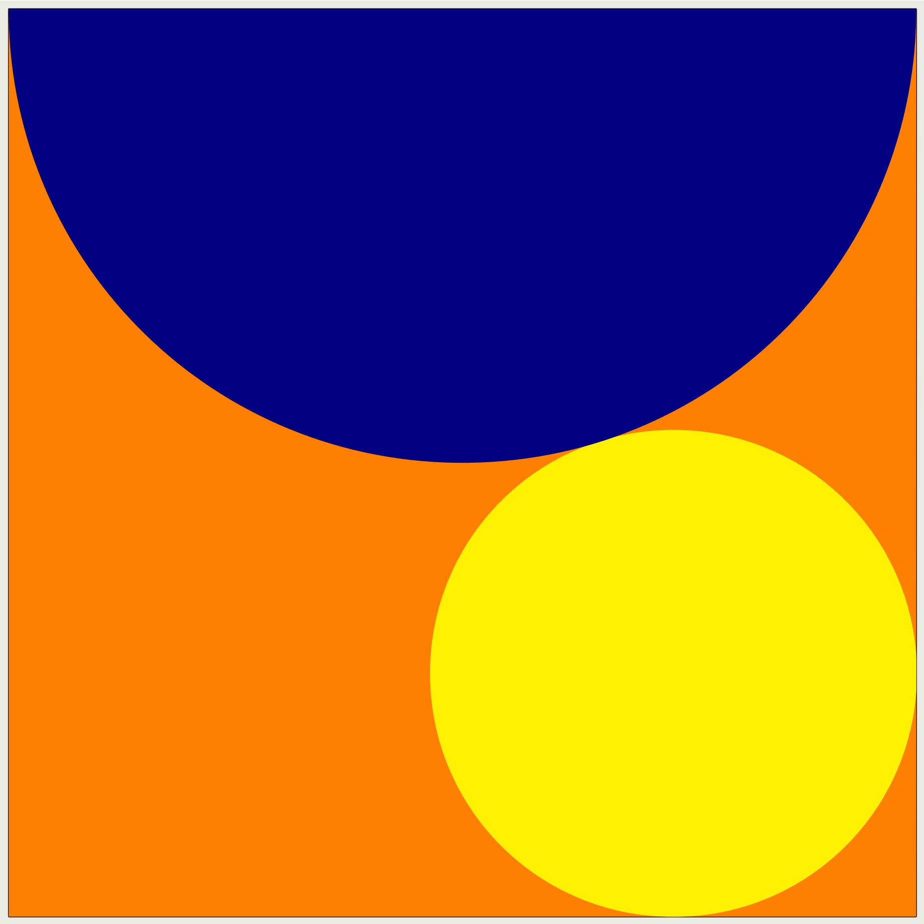
from which r is found :

$$r = \frac{2a}{3 + 2\sqrt{2} + \sqrt{5}}$$

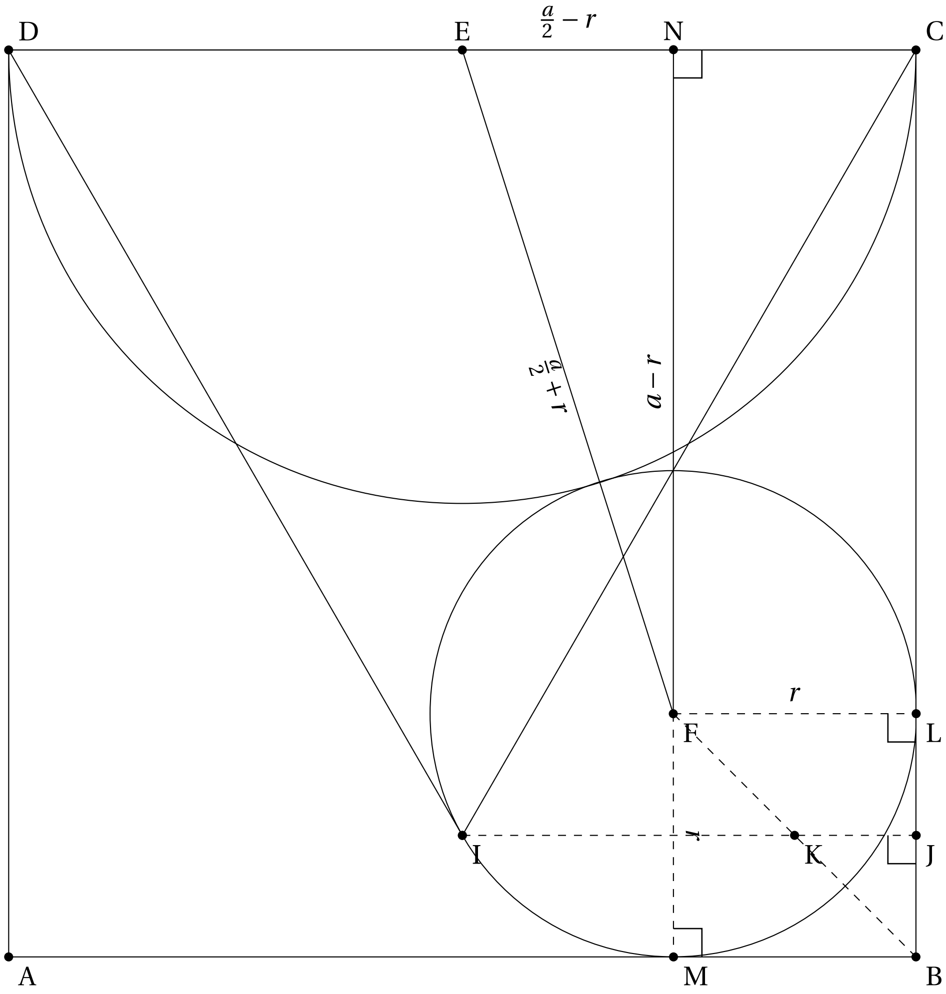
Sangaku in a square - IV

Find a relationship between the radius of the yellow circle and the side of the square.

Example n° 5 Sangaku in a square - IV



```
\begin{tikzpicture}[scale = 1.5]
   \tkzInit
   \tkzPoint*(0,0){A} \tkzPoint*(8,0){B}
   \tkzSquare*(A,B){C}{D}
   \tkzPolygon(B,C,D,A)
   \path[clip] (B)--(C)--(D)--(A)--cycle;
   \tkzPoint*(4,8){F}
   \tkzTrEqui(C,D){I}
   \tkzDrawPoint(I)
   \tkzProjection*(B,C)(I/J)
   \tkzInterLL*(D,B)(I,J){K}
   \tkzCSym*(K)(B/M)
   \tkzCircle(M,I)
   \tkzMathLength(M,I)
   \tkzFillPolygon[color = orange](A,B,C,D)
   \tkzFillCircle[color = yellow](M,\tkzmathLen pt)
   \tkzFillCircle[color = blue!50!black](F,4 cm)%
\end{tikzpicture}
```



Proof :
FNE is a right triangle with hypotenuse EF. We have, $EN^2 + NF^2 = EF^2$ by the Pythagorean theorem. In terms of a and r, the theorem appears as

$$\left(\frac{a}{2} - r\right)^2 + (a - r)^2 = \left(\frac{a}{2} + r\right)^2$$

which is equivalent to

$$4ar = a^2 + r^2$$

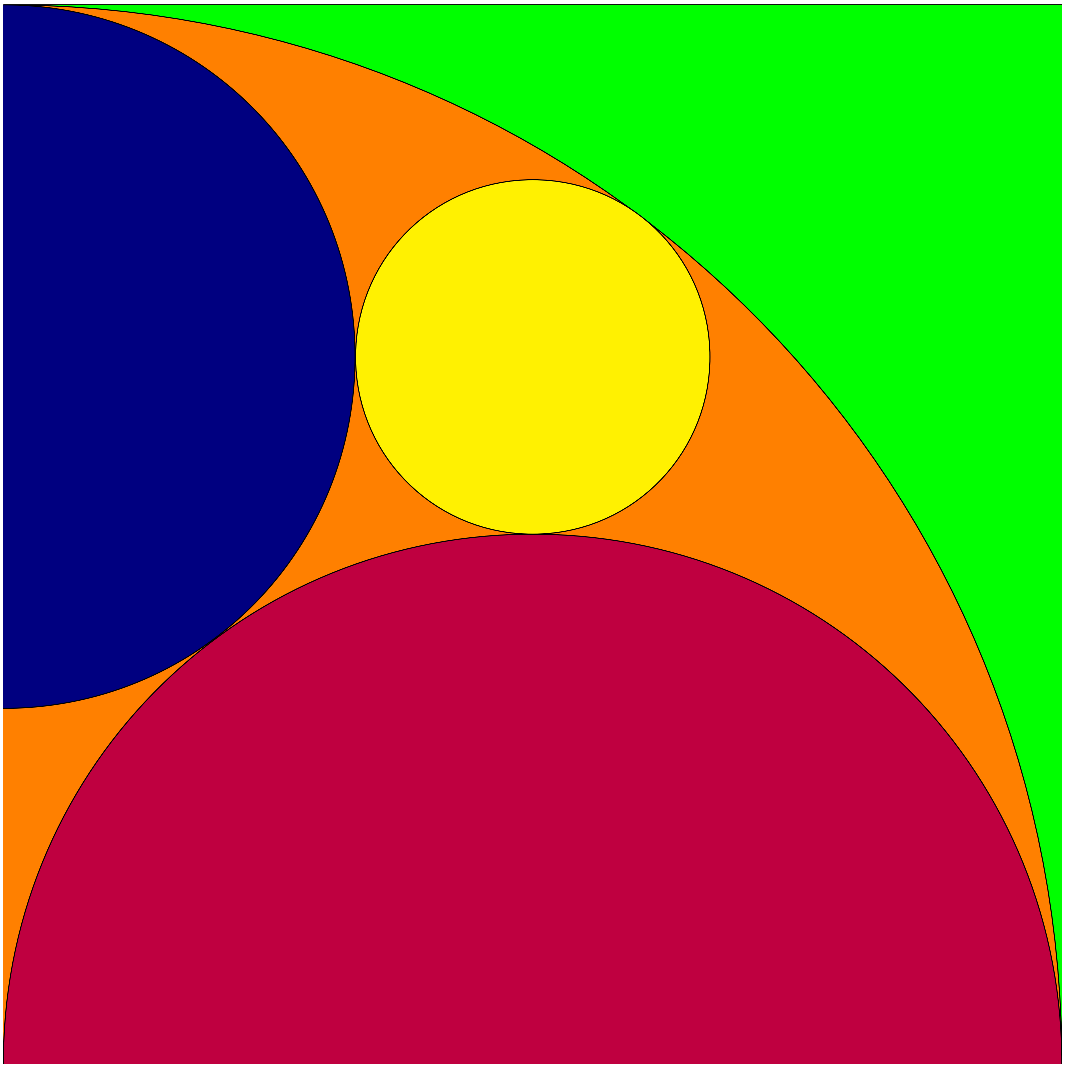
And finally

$$r = a(2 - \sqrt{3})$$

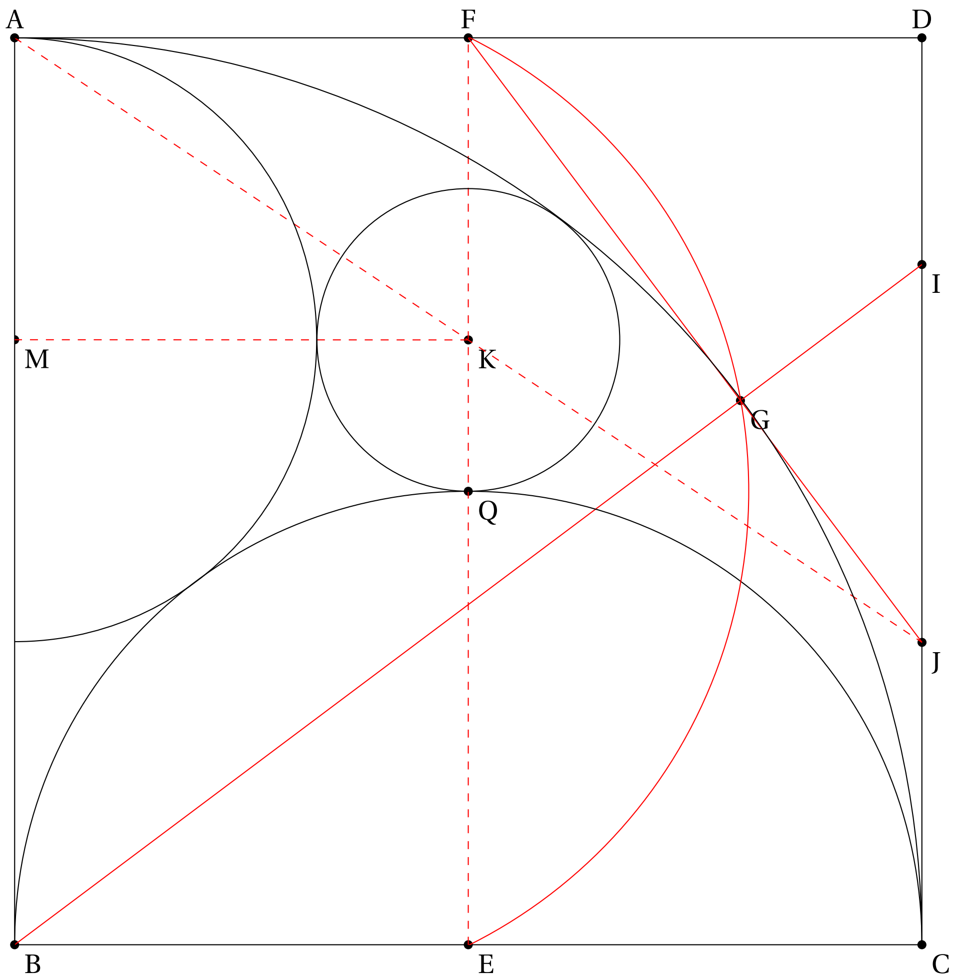
Sangaku in a square - V

This sangaku requires to determine the relative radii of the circles shown can be solved by an application of the Pythagorean theorem. Find a relationship between the radius of the circle and the side of the square.

Example n° 6 Sangaku in a square - V



```
\begin{tikzpicture}[scale = 1.75]
   \tkzInit[xmax = 8] \tkzClip
   \tkzPoint*(0,0){B} \tkzPoint*(8,0){C}
   \tkzPoint*(0,8){A} \tkzPoint*(8,8){D}
   \tkzPolygon(B,C,D,A)
   \path[clip] (B)--(C)--(D)--(A)--cycle;
   \tkzPoint*(4,8){F}\tkzPoint*(4,0){E}\tkzPoint*(4,4){Q}
   \tkzTgtFromP(F,F,A)(B){G}{H}
   \tkzInterLL*(F,G)(C,D){J}
   \tkzInterLL*(A,J)(F,E){K}
   \tkzProjection*(B,A)(K/M)
   \tkzFillPolygon[color = green](A,B,C,D)
   \tkzCircle[style = {fill = orange}](B,A)
   \tkzCircle[style = {fill = blue!50!black}](M,A)
   \tkzCircle[style = {fill = purple}](E,B)
   \tkzCircle[style = {fill = yellow}](K,Q)
\end{tikzpicture}
```



Assume the radius AM equal r and the side of the square is a.

step 1. Firstly, $\mathrm{ME}^2 = \mathrm{BE}^2 + \mathrm{BM}^2$ and the two circles are tangent if $\mathrm{ME} = r + \frac{a}{2}$. The equality becomes

$$\left(r + \frac{a}{2}\right)^2 = \left(\frac{a}{2}\right)^2 + (a - r)^2$$

$$r = \frac{a}{3}$$

we have

$$\mathrm{ME} = \mathrm{BK} = \frac{5}{6}a$$

step 2.

$$\mathrm{KQ} = \mathrm{KE} - \mathrm{QE} = \frac{2a}{3} - \frac{a}{2} = \frac{a}{6}$$

and

$$\mathrm{MK} - \frac{a}{3} = \frac{a}{2} - \frac{a}{3} = \frac{a}{6}$$

The circle K is tangent mutually at the circle E and the circle M.

step 3. The little circle with center K is tangent at the circle B.

$$a - \frac{5a}{6} = \frac{a}{6}$$

step 4. A goog method to obtain K is finally to place I such as DI = $\frac{a}{4}$. Therefore, BI intercepts the big circle in G with GI = $\frac{a}{4}$,FG = $\frac{a}{2}$ and FG orthogonal to BG. FG intercepts BC in J such as DJ = $\frac{2a}{3}$. K is the common point between AJ and EF.

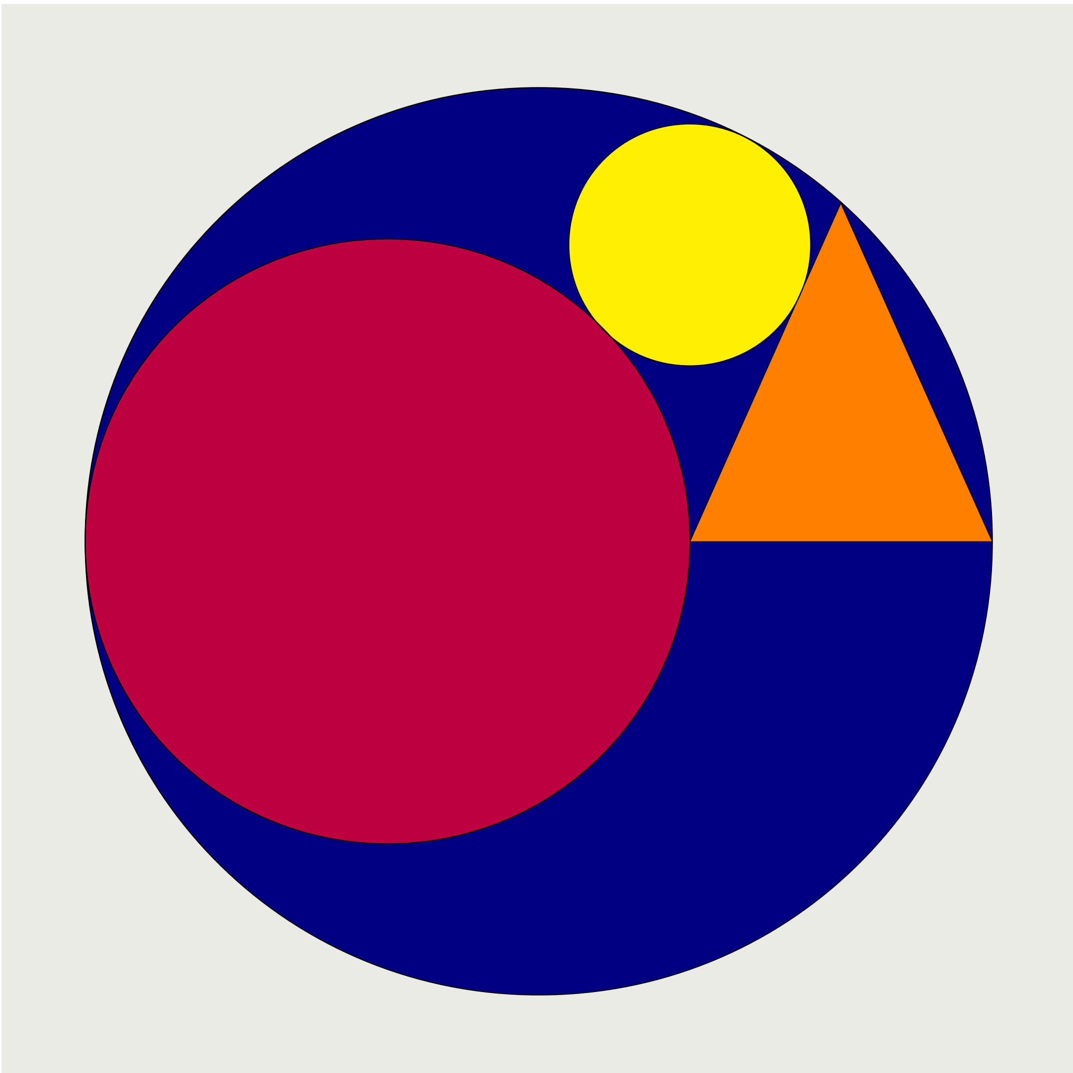
Sangaku - Circle Inscribing

Construct the figure consisting of a circle centered at O, *a second smaller circle centered at* O_2 *tangent to the first, and an isosceles triangle whose base* [AB] *completes the diameter of the larger circle* [XB] *through the smaller* [XA]. *Now inscribe a third circle with center* O_3 *inside the large circle, outside the small one, and on the side of a leg of the triangle.*

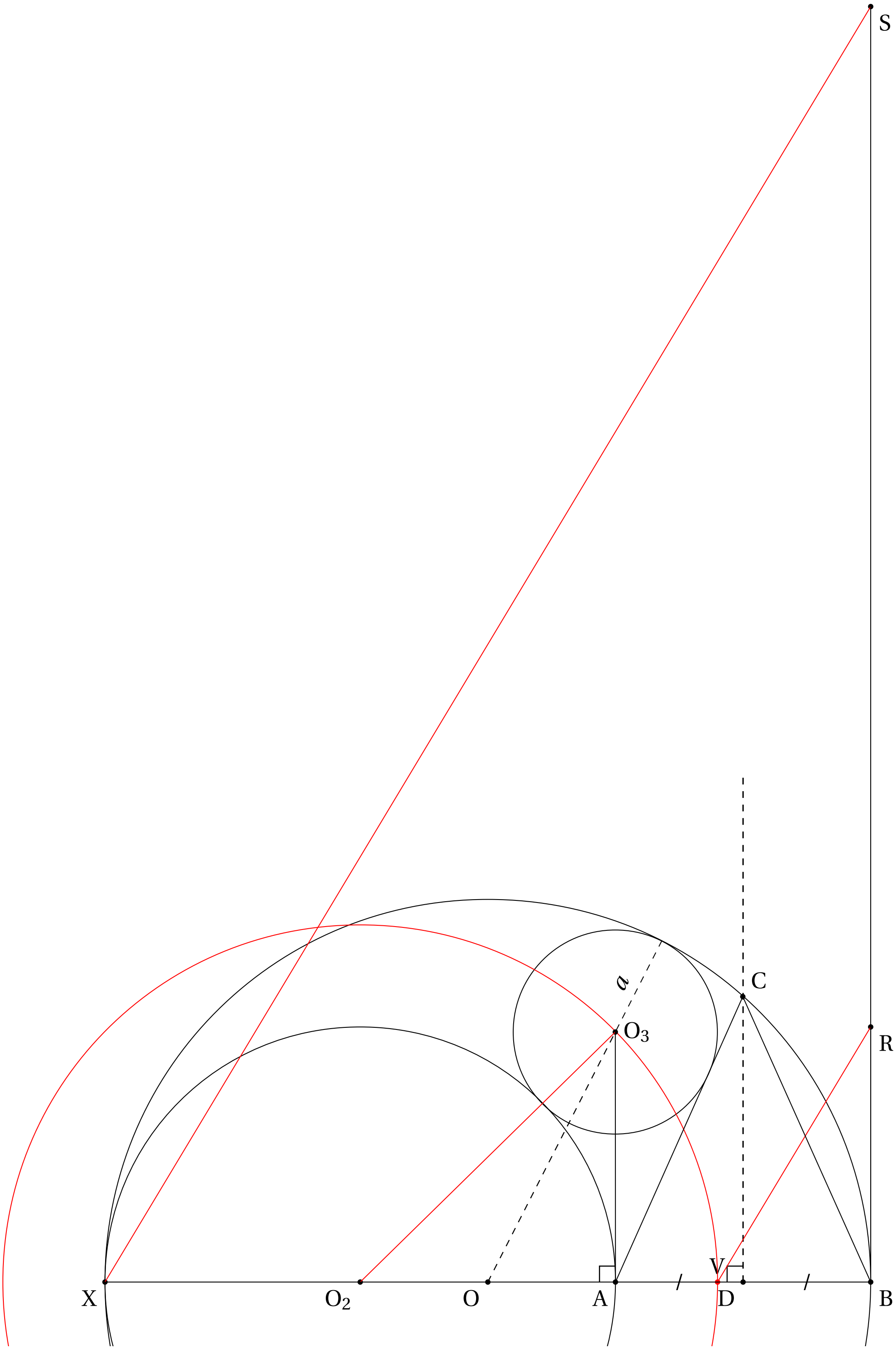
Example n° 7 Circle Inscribing

References
Weisstein, Eric W. "Circle Inscribing." From MathWorld–A Wolfram Web
http://mathworld.wolfram.com/CircleInscribing.html
Alexander Bogomolny
http://www.cut-the-knot.org/

In this problem, from an 1803 sangaku found in Gumma Prefecture, the base of an isosceles triangle sits on a diameter of the large blue circle. This diameter also bisects the purple circle, which is inscribed so that it just touches the inside of the blue circle and one vertex of the orange triangle, as shown. The yellow circle is inscribed so that it touches the outsides of both the purple circle and the triangle, as well as the inside of the blue circle. A line segment connects the center of the yellow circle and the intersection point between the purple circle and the orange triangle. Show that this line segment is perpendicular to the drawn diameter of the blue circle.



```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \xdef\ORadius{6}
   \xdef\OORadius{4}
   \pgfmathparse{(2*(\ORadius-\OORadius))/(\ORadius/\OORadius+1)}%
   \global\let\OOORadius\pgfmathresult%
   \pgfmathparse{\ORadius-\OOORadius}%
   \global\let\OOOORadius\pgfmathresult%
   \pgfmathparse{2*\OORadius-\ORadius}%
   \global\let\XA\pgfmathresult%
   \tkzPoint*(0,0){O}
   \ifdim\XA pt  =  0pt\relax%
      \tkzPoint[pos = below right](\XA,0){A}
   \else%
      \tkzPoint*(\XA,0){A}
   \fi%
   \tkzPoint*(\OORadius,0){D}
   \tkzPoint*(-\ORadius,0){X}
   \tkzPoint*(\ORadius,0){B}
   \tkzPoint*(\OORadius-\ORadius,0){O2}
   \tkzMediatorLine*[prefix = m](A,B)
   \tkzInterLCR(ml,mr)(O,\ORadius cm){C}{E}
   \tkzLineOrth*[prefix = p](X,A)(A)
   \ifdim\XA pt < 0 pt\relax%
      \tkzInterLCR(pl,pr)(O,\OOOORadius cm){O4}{O3}
   \else%
      \tkzInterLCR(pr,pl)(O,\OOOORadius cm){O3}{O4}
   \fi%
   \tkzInterLCR(O,O3)(O,\ORadius cm){W}{Z}
   \tkzFillCircle[color = blue!50!black](O,\ORadius cm)%
   \tkzFillPolygon[color = orange](A,B,C)%
   \tkzFillCircle[color = yellow](O3,\OOORadius cm)%
   \tkzFillCircle[color = purple](O2,\OORadius cm)%
   \tkzSegment(C/B,C/A)
   \tkzCircleR(O,\ORadius)
   \tkzCircleR(O2,\OORadius)
   \tkzCircleR(O3,\OOORadius)
\end{tikzpicture}
```


S
C
R
a
O_3
V
X
O_2
O
A
D
B

To find the explicit position and size of the circle, let the circle with center O have radius R and be centered at O and let the circle with center O_2 have radius r.

$$(r+a)^2 = r^2 + y^2$$

$$(\mathrm{R}-a)^2 = (\mathrm{R}-2r)^2 + y^2$$

for a and y gives

$$a = 2r\frac{\mathrm{R}-r}{\mathrm{R}+r}$$

$$y = \mathrm{AO}_3 = 2\sqrt{2\mathrm{R}}r\frac{\sqrt{\mathrm{R}-r}}{\mathrm{R}+r}$$

but now we need to prove that the circle is tangent to the line AC.
Let α the angle ACD and the angle O_3AC

$$\sin(\alpha) = \frac{\mathrm{AD}}{\mathrm{AC}}$$

$\mathrm{OD} = r$ and $\mathrm{AD} = \mathrm{R} - r$
OCD is a right triangle with hypotenuse OC = R. We have, $\mathrm{OD}^2 + \mathrm{CD}^2 = \mathrm{OC}^2$ by the Pythagorean theorem. In terms of r, the theorem appears as

$$r^2 + \mathrm{CD}^2 = \mathrm{R}^2$$

which is equivalent to

$$\mathrm{CD}^2 = \mathrm{R}^2 - r^2$$

and with the right triangle ADC and the Pythagorean theorem

$$\mathrm{AC}^2 = \mathrm{AD}^2 + \mathrm{CD}^2 = (\mathrm{R}-r)^2 + \mathrm{R}^2 - r^2 = 2\mathrm{R}(\mathrm{R}-r)$$

finally

$$\sin(\alpha) = \frac{\mathrm{AD}}{\mathrm{AC}} = \frac{\mathrm{R}-r}{\sqrt{2\mathrm{R}(\mathrm{R}-r)}} = \frac{\sqrt{\mathrm{R}-r}}{\sqrt{2\mathrm{R}}}$$

Let H the projection point of O_3 on the line AC, and d the length of O_3H

$$\sin(\alpha) = \frac{\mathrm{O_3H}}{\mathrm{AO_3}} = \frac{d}{y} = \frac{d}{2\sqrt{2\mathrm{R}}r\frac{\sqrt{\mathrm{R}-r}}{\mathrm{R}+r}}$$

Using the two forms of sin(α)

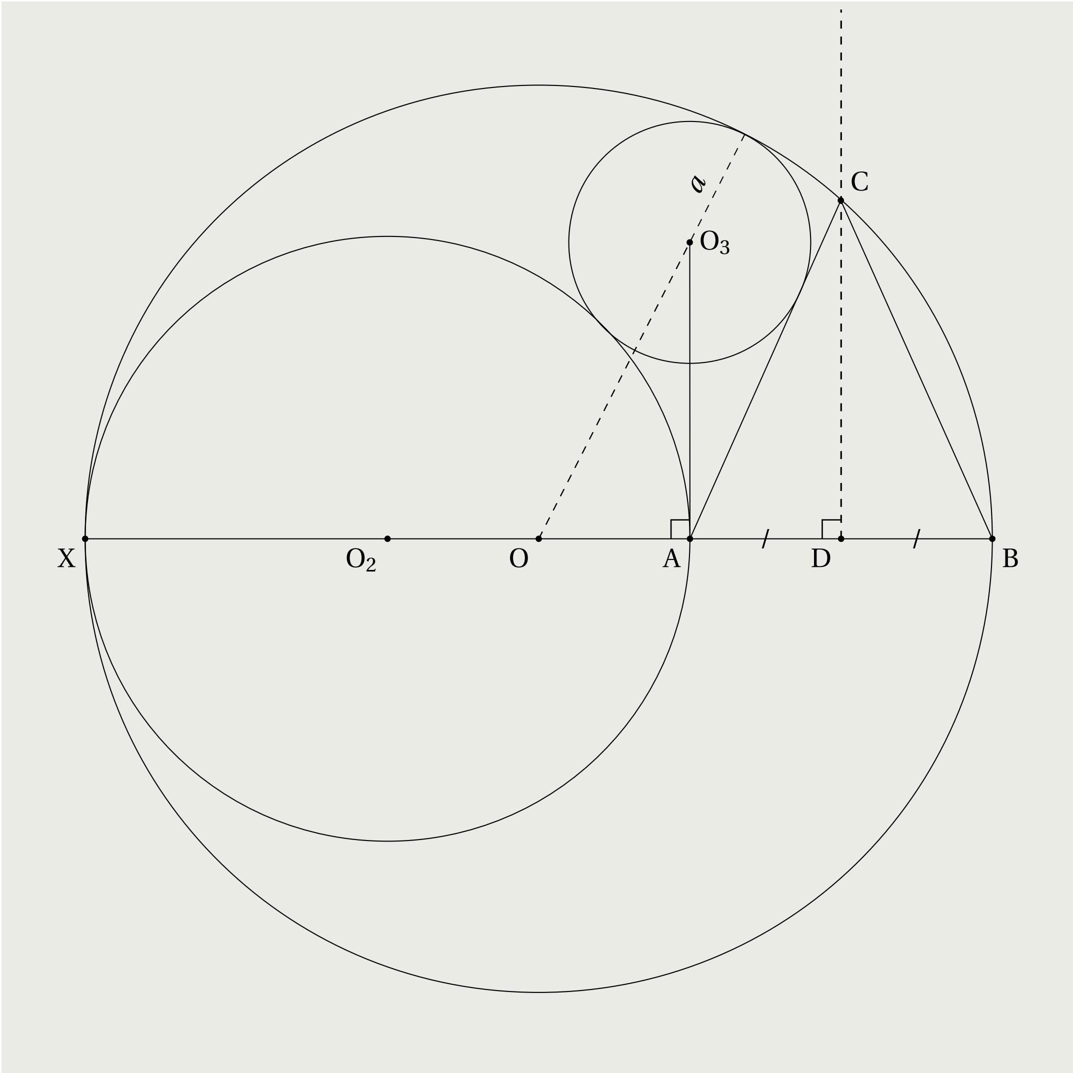
$$\frac{d}{2\sqrt{2\mathrm{R}}r\frac{\sqrt{\mathrm{R}-r}}{\mathrm{R}+r}} = \frac{\sqrt{\mathrm{R}-r}}{\sqrt{2\mathrm{R}}}$$

So

$$d = 2r\frac{\mathrm{R}-r}{\mathrm{R}+r} = a$$

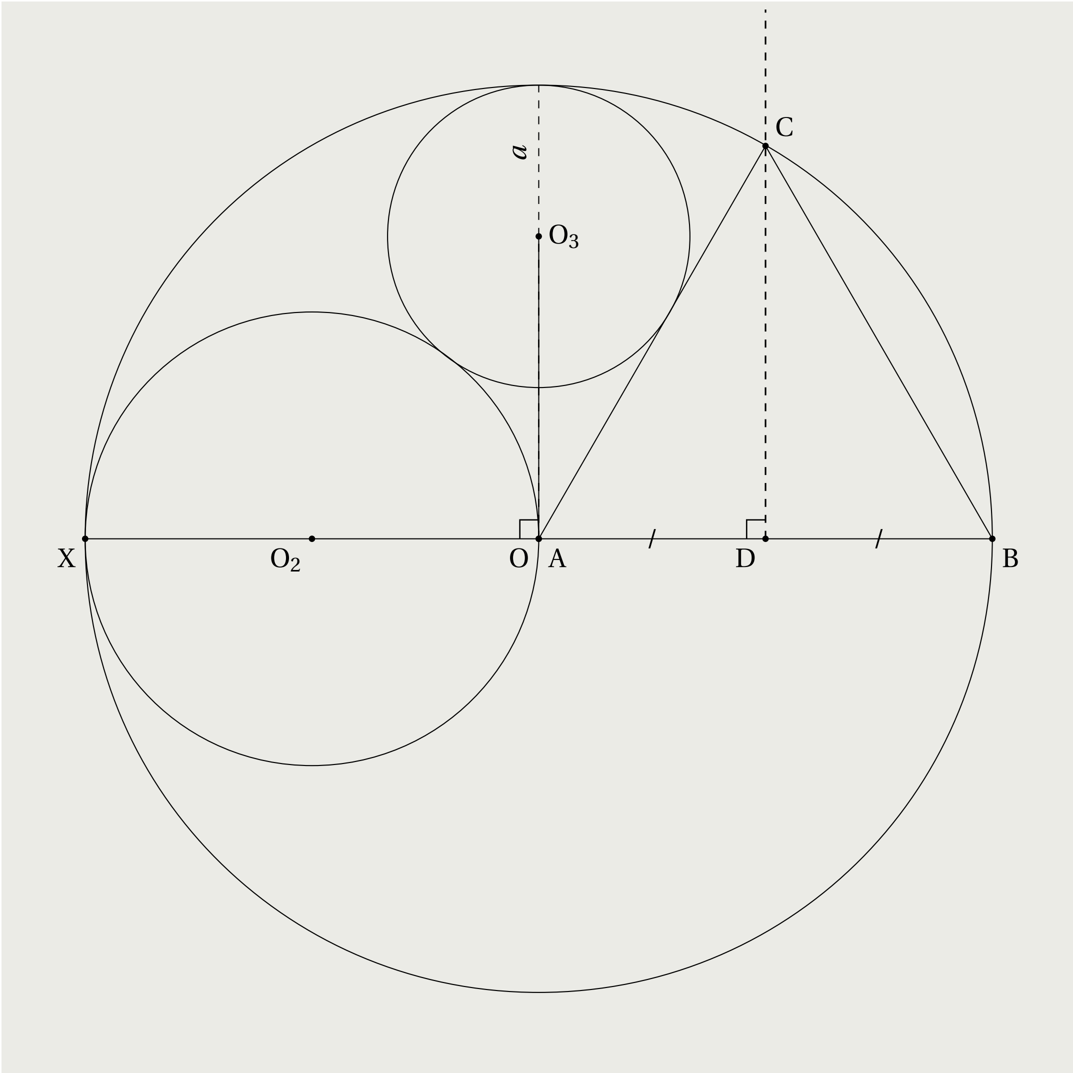
The code on the page is needed to get the examples.

```
\newcommand*{\CircleInscribing}[2]{%
    \xdef\ORadius{#1}
    \xdef\OORadius{#2}
    \pgfmathparse{(2*(\ORadius-\OORadius))/(\ORadius/\OORadius+1)}%
    \global\let\OOORadius\pgfmathresult%
    \pgfmathparse{\ORadius-\OOORadius}%
    \global\let\OOOORadius\pgfmathresult%
    \pgfmathparse{2*\OORadius-\ORadius}%
    \global\let\XA\pgfmathresult%
     \tkzPoint[pos = below left](0,0){O}
    \ifdim\XA pt  =  0pt\relax%
        \tkzPoint[pos = below right](\XA,0){A}
    \else%
        \tkzPoint[pos = below left](\XA,0){A}
    \fi%
    \tkzPoint[pos = below left](\OORadius,0){D}
    \tkzPoint[pos = below left](-\ORadius,0){X}
    \tkzPoint[pos = below right](\ORadius,0){B}
    \tkzPoint[name = $O_2$,pos = below left](\OORadius-\ORadius,0){O2}
    \tkzSegmentMark[symbol = /](D/B,D/A)
    \tkzCircleR(O,\ORadius)
    \tkzCircleR(O2,\OORadius)
    \tkzMediatorLine[prefix = m,kr = 2,kl = 0,style = dashed](A,B)
    \tkzInterLCR(ml,mr)(O,\ORadius cm){C}{E}
    \tkzLineOrth*[prefix = p](X,A)(A)
    \ifdim\XA pt < 0 pt\relax%
        \tkzInterLCR(pl,pr)(O,\OOOORadius cm){O4}{O3}
    \else%
        \tkzInterLCR(pr,pl)(O,\OOOORadius cm){O3}{O4}
    \fi%
   \tkzRightAngle(X/D/C,X/A/O3)
   \tkzCircleR(O3,\OOORadius)
   \tkzDrawPoint[name = $O_3$,pos = right](O3)
   \tkzDrawPoint[pos = above right](C)
   \tkzSegment[style = dashed](O/O3)
   \tkzSegment(A/O3,C/B,C/A,X/B)
   \tkzInterLCR(O,O3)(O,\ORadius cm){W}{Z}
   \tkzSegment[label = $a$,time = .85,style = dashed](O/Z)
}%
```



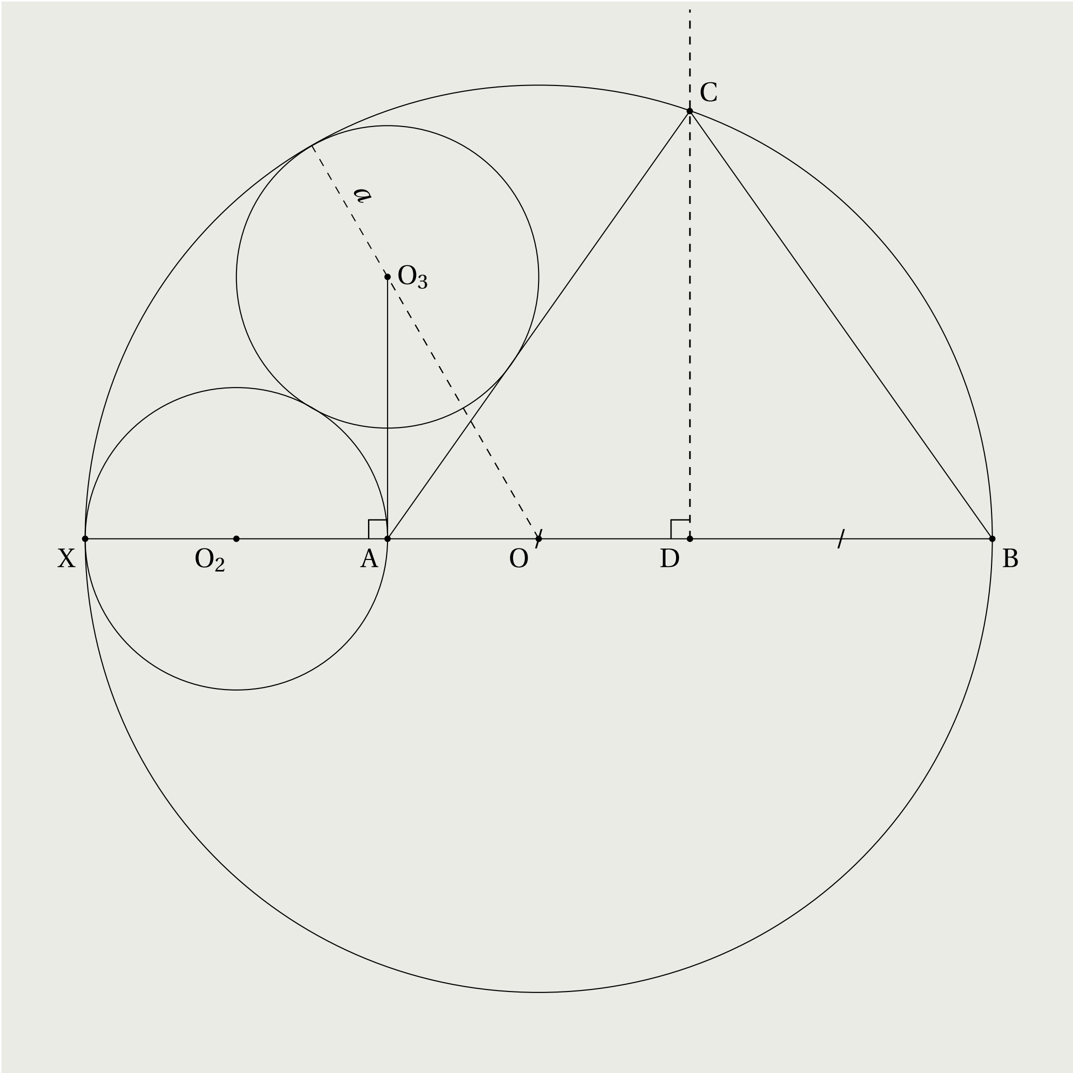

```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{4}
\end{tikzpicture}
```

FIG. 6: Sangaku problem (1803) : R1 = 6 cm et R2 = 4 cm



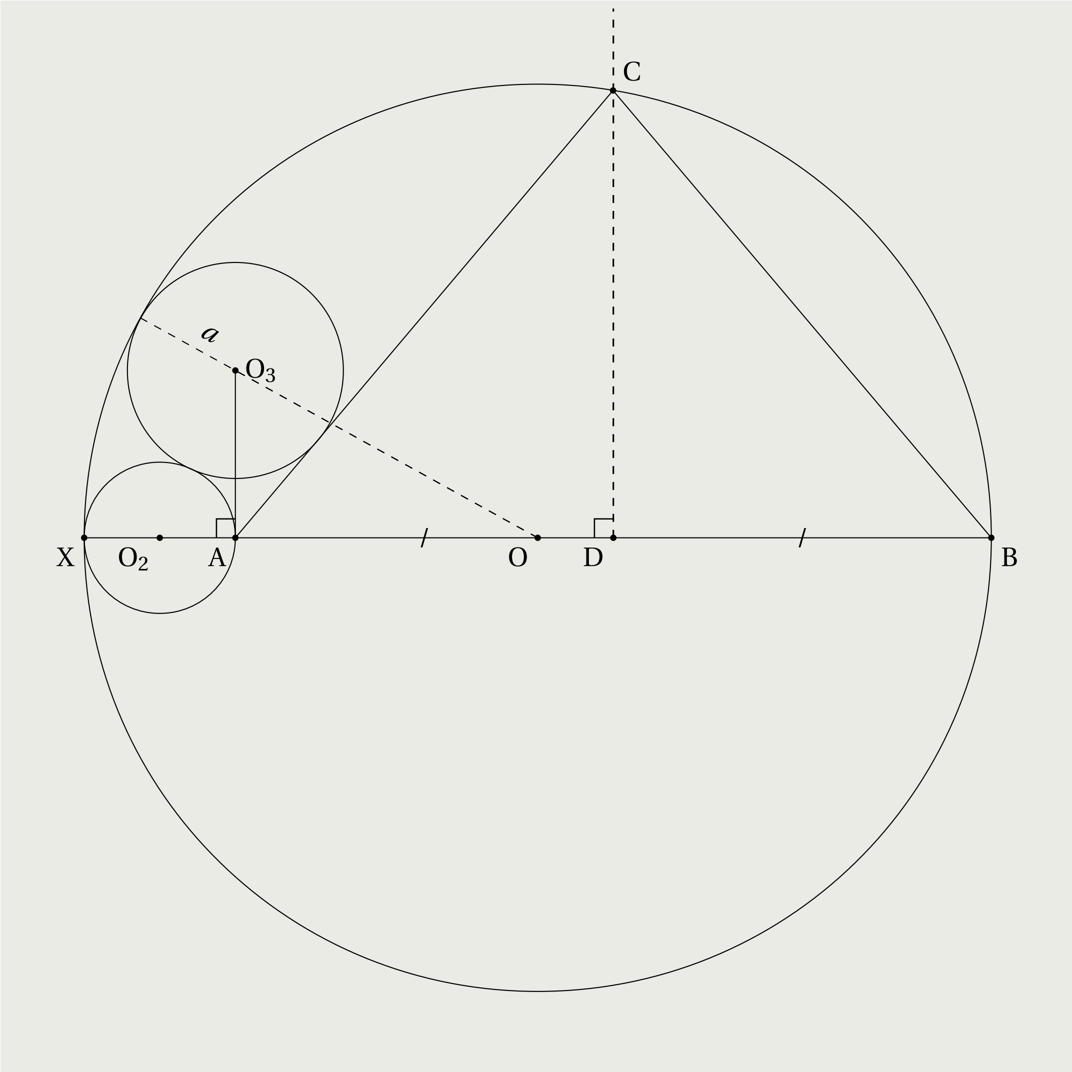

```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{3}
 \end{tikzpicture}
```

FIG. 7: Sangaku problem (1803) : R1 = 6 cm et R2 = 3 cm




```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{2}
\end{tikzpicture}
```

FIG. 8: Sangaku problem (1803) : R1 = 6 cm et R2 = 2 cm




```
\begin{tikzpicture}[scale = 1]
   \tkzInit[xmin = -7,xmax = 7,ymin = -7,ymax = 7]
   \tkzClip
   \CircleInscribing{6}{1}
\end{tikzpicture}
```

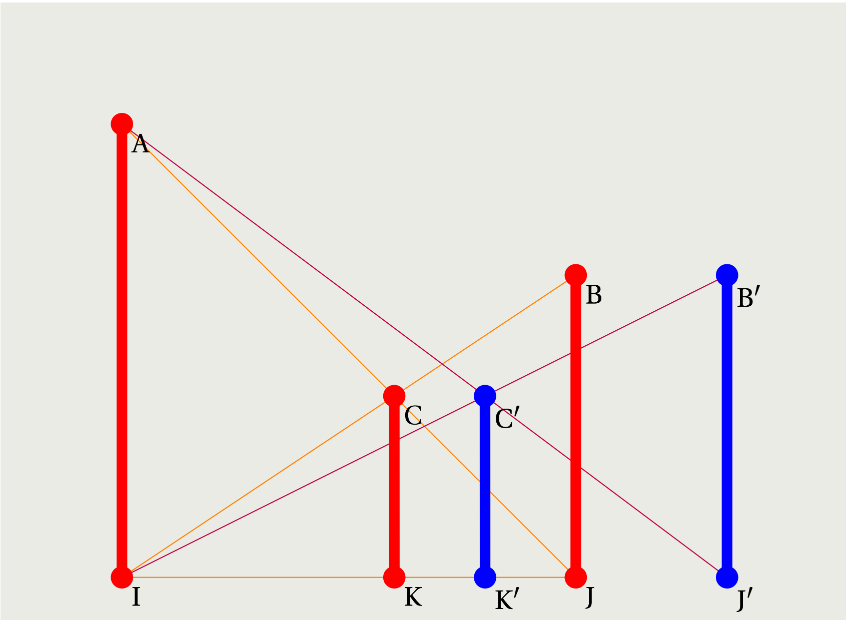
FIG. 9: Sangaku problem (1803) : R1 = 6 cm et R2 = 1 cm

Sangaku - Harmonic mean

Two vertical segments AI, BJ *(*AI = a *and* BJ = b*), the intersection* C *of the diagonals is at the height that depends solely on* AI *and* BJ. *In fact*

$$\frac{1}{\mathrm{CK}} = \frac{1}{\mathrm{AI}} + \frac{1}{\mathrm{BJ}} = \frac{1}{a} + \frac{1}{b}$$

Example n° 8 Harmonic mean




```
\begin{tikzpicture}
  \tkzInit[xmin = -1,xmax = 9,ymax = 7]\tkzClip[space=.5]
  \tkzPoint(0,6){A}\tkzPoint(0,0){I}
  \tkzPoint(6,4){B}\tkzPoint(8,4){B'}
  \tkzPoint(6,0){J}\tkzPoint(8,0){J'}
  \tkzInterLL*(A,J)(B,I){C}
  \tkzProjection(I,J)(C/K)
  \tkzInterLL(A,J')(B',I){C'}
  \tkzProjection(I,J)(C'/K')
  \tkzSegment[color = orange](A/J,B/I,I/J)
  \tkzSegment[color = purple](A/J',B'/I)
  \tkzSegment[lw = 4pt,color = red](C/K,A/I,B/J)
  \tkzSegment[lw = 4pt,color = blue](C'/K',B'/J')
  \tkzDrawPoint[color = red,size = 4pt](A,I,C,K,B,J)
  \tkzDrawPoint[color = blue,size = 4pt](C',K',B',J')
\end{tikzpicture}
```

Let's denote AI = a, BJ = b, CK = c, IK = α and KJ = β.
Then from similar triangles AIJ and IBJ we have the proportion

$$\frac{c}{a} = \frac{\beta}{\alpha+\beta} \text{ and } \frac{c}{b} = \frac{\alpha}{\alpha+\beta}$$

We can add the two equalities

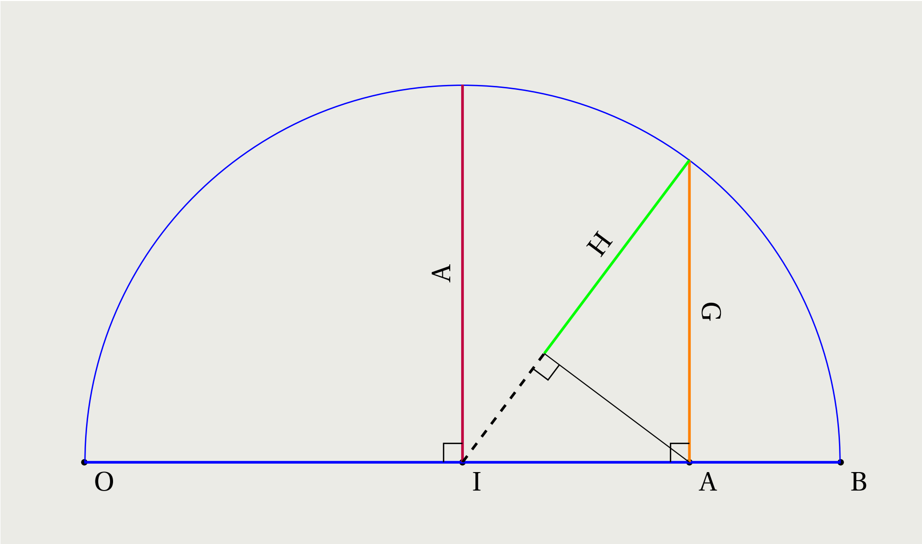
$$\frac{c}{a} + \frac{c}{b} = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} = 1$$

A division by c gives the desired result :

$$\frac{1}{a} + \frac{1}{b} = \frac{1}{c}$$

which says that c is the double of the harmonic mean of a and b.

Remark : a and b two numbers such as OA = a and AB = b




```
\begin{tikzpicture}
   \tkzInit[xmin = -1,ymin = -1,xmax = 11,ymax = 6]\tkzClip
   \tkzPoint(0,0){O}\tkzPoint(8,0){A}%
   \tkzPoint(10,0){B}\tkzPoint(5,0){I}
   \tkzPoint*(5,5){K}
   \tkzSegment[color = blue,lw = 1pt](O/B)
   \tkzClipSector(I,5 cm)(0,180)
   \tkzCircleR[color = blue,lw = 1pt](I,5 cm)
   \tkzLineOrth[prefix = h](O,B)(A)
   \tkzInterLCR(A,hr)(I,5 cm){G}{G'}
   \tkzSegment[lw = 1pt,color = purple](I/K)
   \tkzSegment[lw = 1pt,color = orange](A/G)
   \tkzProjection*(I,G)(A/H)
   \tkzSegment[lw = 1pt,color = green](H/G)
   \tkzSegment[lw = 1pt,style = dashed](I/H)
   \tkzRightAngle(O/I/K,O/A/G,A/H/I)
   \tkzSegment(A/H)
   \tkzSegmentMark[label = $A$](I/K)
   \tkzSegmentMark[label = $G$](A/G)
   \tkzSegmentMark[label = $H$](H/G)
\end{tikzpicture}
```

It is easy to find

$$\frac{\mathrm{H}}{\mathrm{G}} = \frac{\mathrm{G}}{\mathrm{A}}$$

In this case, we have $\mathrm{G}^2 = \mathrm{A} \times \mathrm{H}$ and $\mathrm{H} = \frac{\mathrm{G}^2}{\mathrm{A}}$
For two numbers a and b, $\mathrm{A} = \frac{a+b}{2}$ and $\mathrm{G} = \sqrt{ab}$, so

$$\frac{1}{\mathrm{H}} = \frac{\mathrm{A}}{\mathrm{G}^2} = \frac{a+b}{2ab}$$

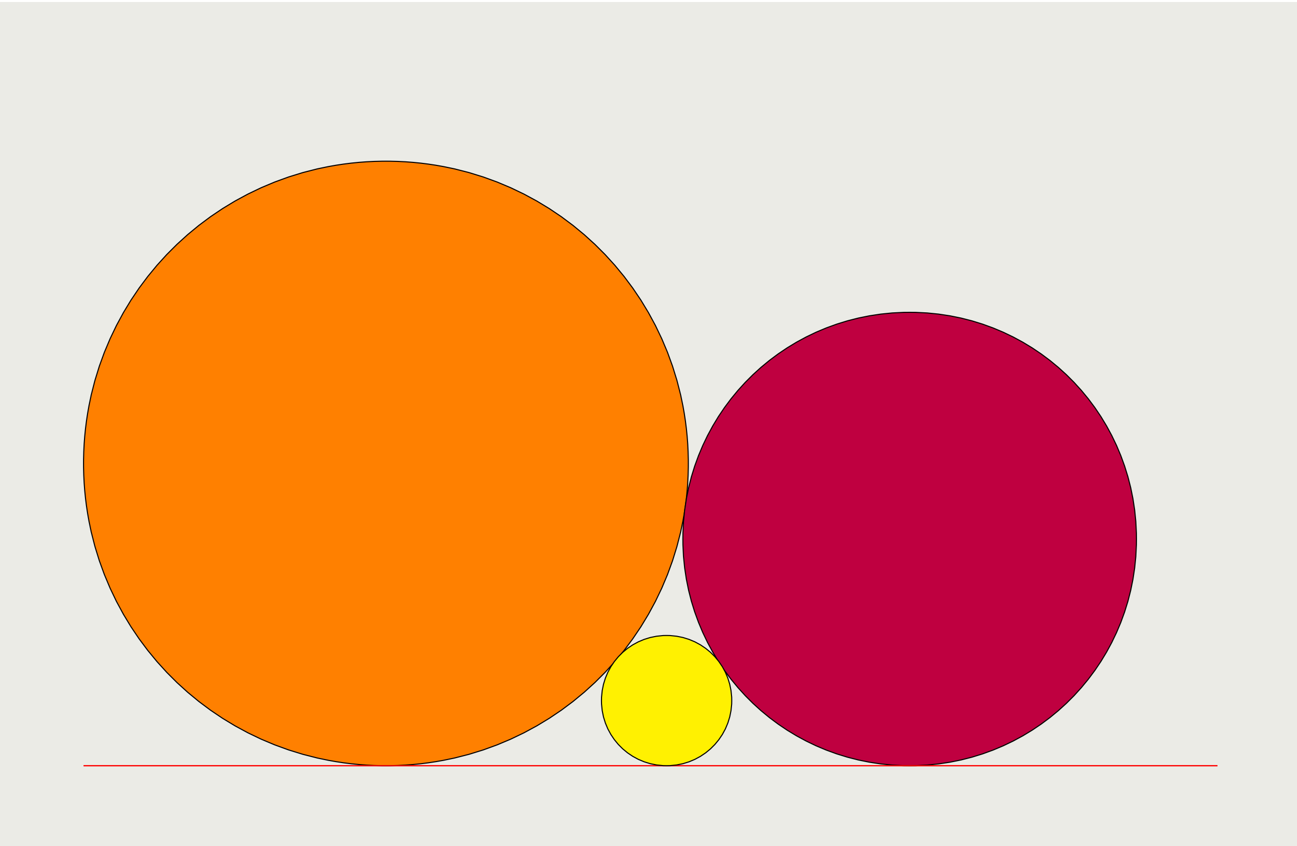
Finally

$$\frac{1}{\mathrm{H}} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right)$$

Sangaku - Three Tangent Circles

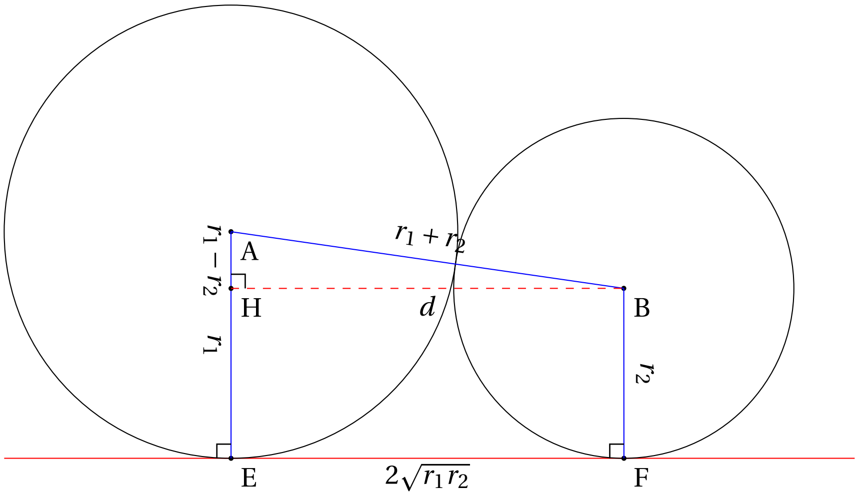
Given three circles tangent to each other and to a straight line, express the radius of the middle circle via the radii of the other two. This problem was given as a Japanese temple problem on a tablet from 1824 in the Gumma Prefecture (MathWorld)

Example n° 9 Three Tangent Circles



```
\begin{tikzpicture}
  \tkzInit[xmin = -1,ymin = -1,xmax = 16]\tkzClip
  \tkzPoint*(4,4){A}\tkzPoint*(10.928,3){B}
  \tkzPoint*(0,0){O}\tkzPoint*(15,0){X}
  \tkzSegment[color = red](O/X)
  \tkzCircleR[style = {fill = orange}](A,4 cm)
  \tkzCircleR[style = {fill = purple}](B,3 cm)
  \pgfmathparse{4+8*sqrt(3)/(2+sqrt(3))}
  \edef\cx{\pgfmathresult}
  \pgfmathparse{12/((2+sqrt(3))*(2+sqrt(3)))}
  \edef\cy{\pgfmathresult}
  \tkzPoint(\cx,\cy){C}
  \tkzSegment[color = red](O/X)
  \tkzCircleR[style = {fill = yellow}](C,\cy cm)
\end{tikzpicture}
```

As the diagram below shows we have three right triangles with the hypotenuses joining the centers of the three circles.
Using x and y to denote the horizontal distances between pairs of the circles, and R, R1, R2 at their radii, the triangles have the following sides :
$(r_1 - r)2 + x^2 = (r_1 + r)^2$ $(r_2 - r)^2 + y^2 = (r_2 + r)^2$ $(r_2 - r_1)2 + (x + y)2 = (r_2 + r_1)^2$. After simplification the equations become
$(x + y)2 = 4r_1 r_2$

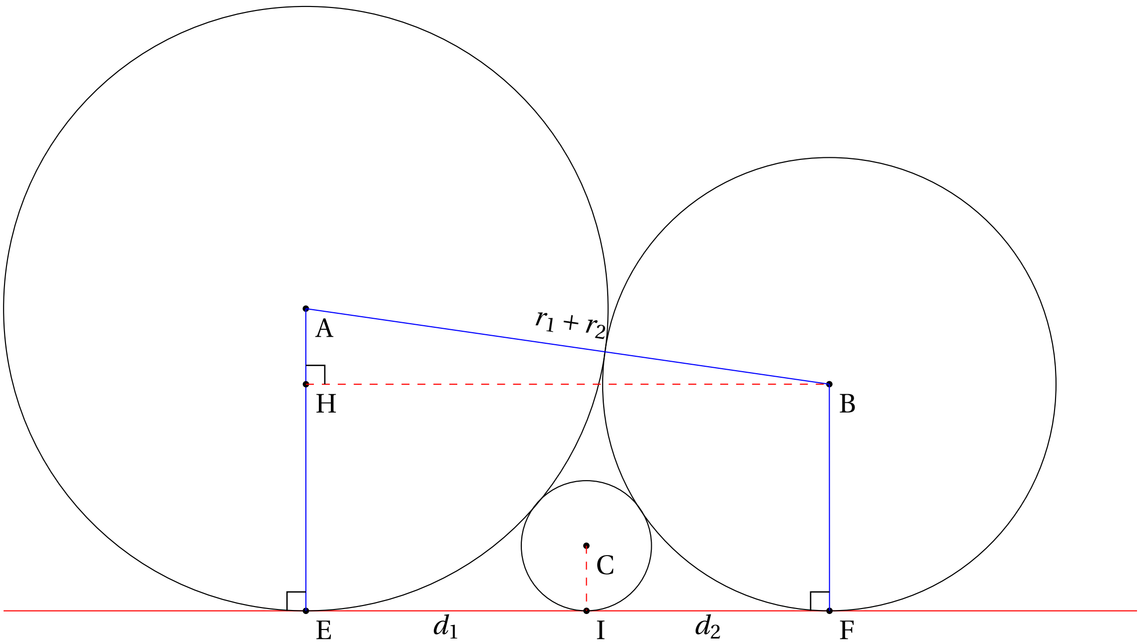


ABH is a right triangle with hypotenuse AB = $r_1 + r_2$. We have, $AH^2 + HB^2 = AB^2 = d^2$ by the Pythagorean theorem.
If two circles A and B and of radii r_1 and r_2 ($r_1 > r_2$) are mutually tangent to each other and a line OX, then their centers are separated by a horizontal distance given by solving

$$d^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2$$

After simplification the equation gives

$$d = 2\sqrt{r_1 r_2}$$



Using d_1 and d_2 to denote the horizontal distances between pairs of the circles, and r, r_1, r_2 at their radii, the triangles have the following sides :

$$2\sqrt{r_1 r_2} = 2\sqrt{r_1 r} + 2\sqrt{r r_2}$$

Divide now by $\sqrt{r}\sqrt{r_1}\sqrt{r_2}$ to obtain

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}}$$

$$\sqrt{r} = \frac{\sqrt{r_1}\sqrt{r_2}}{\sqrt{r_1}+\sqrt{r_2}}$$

$$d_1 = 2\sqrt{rr_1} = 2\sqrt{r_1}\frac{\sqrt{r_1}\sqrt{r_2}}{\sqrt{r_1}+\sqrt{r_2}} = \frac{2r_1\sqrt{r_2}}{\sqrt{r_1}+\sqrt{r_2}}$$

How to make this construction with a ruler and a compass

step 1. We want to draw two circles with centers A and B of radii $r_1 = 7cm$ and $r_2 = 4cm$ mutually tangent to each other and a line OX. OA = $7cm$, OP = $4cm$ and P′ is the symmetric point of p relatively the center O.

step 2. The circle with diameter AP′ intercepts the OX axis in a point I such as the length OI = $\sqrt{ab}$. It is easy to obtain H such OH = $\sqrt{ab}$. B is a point such as OHBP is a rectangle. We can draw the circle with center B and radius BH.

step 3. Now we can use the sangaku about harmonic mean. The line OB intercepts the line AH in a point K such as

$$\frac{1}{\mathrm{KJ}} = \frac{1}{\mathrm{OA}} + \frac{1}{\mathrm{HB}} = \frac{1}{a} + \frac{1}{b}$$

$$\frac{1}{\sqrt{r}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}$$

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b} + \frac{2}{\sqrt{ab}}$$

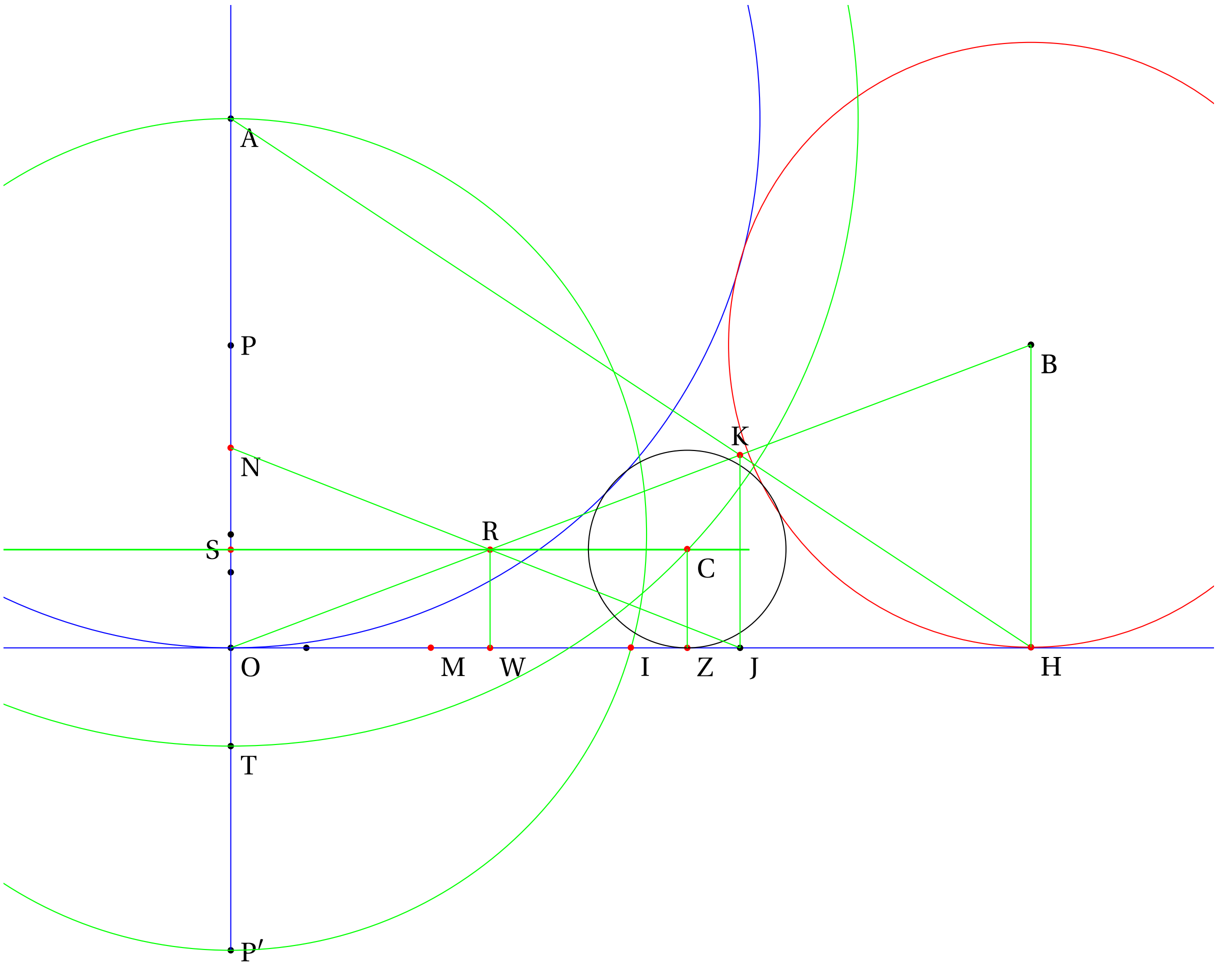
$$\frac{1}{r} = \frac{1}{\mathrm{KJ}} + \frac{1}{\frac{\sqrt{ab}}{2}}$$

To find r , we need only to represent $\frac{\sqrt{ab}}{2}$.

OI = $\sqrt{ab}$, OM = $\frac{\sqrt{ab}}{2}$ and ON = OM

step 4. The line NJ intercepts the line OK in a point R, if W is the projection of R on OX axis, we have RW = r. Let S the projection point of R on OY axis.

step 5. The point C is the intersection of the circle with center A and radius $a + r$ and the line (SR).




```
\begin{tikzpicture}
  \tkzInit[xmin = -3,ymin = -5,xmax = 13,ymax = 8.5]\tkzClip
  \tkzPoint(0,7){A}\tkzPoint(0,0){O}
  \tkzPoint[pos = right](0,4){P}\tkzPoint[pos = right](0,-4){P'}
  \tkzPoint[noname](0,1){V}\tkzPoint[noname](1,0){U}
  \tkzPoint*(16,0){X}\tkzPoint*(-5,0){X'}
  \tkzPoint*(0,16){Y}\tkzPoint*(0,-4){Y'}
  \tkzSegment[color = blue](X'/X,Y'/Y)
  \tkzCircleR[color = blue](A,7 cm)\tkzCircle*[color = green](A,P')
  \tkzMidPoint[noname](A,P'){pt1}
  \tkzInterLCR(O,X)(pt1,5.5 cm){BB}{I}\tkzCSym(I)(O/H)
  \tkzTranslation[color = red](O,H)(P/B)\tkzCircle[color = red](B,H)
  \tkzSegment[color = green](A/H,O/B)
  \tkzInterLL[color = red,pos = above](A,H)(O,B){K}
  \tkzProjection(O,X)(K/J)\tkzMidPoint[color = red](O,I){M}
  \tkzRotate[color = red](O,90)(M/N)
  \tkzInterLL[color = red,pos = above](N,J)(O,B){R}
  \tkzProjection[color = red,pos = left](O,A)(R/S)
  \tkzCSym(O)(S/T)\tkzCircle[color = green](A,T)
  \tkzLine[color = green](R/S)
  \tkzInterLCR(S,R)(A,\tkzmathLen pt){C'}{C}
  \tkzDrawPoint[color = red](C,I,H)\tkzProjection[color = red](O,X)(C/Z,R/W)
  \tkzCircle(C,Z)\tkzSegment[color = green](K/J,B/H,N/J,C/Z,R/W)
\end{tikzpicture}
```